ASRL TR 164-2

DETAILED EXTENSIONS OF PERTURBATION METHODS FOR NONLINEAR PANEL FLUTTER

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Errata for ASRL TR 164-2

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Page 2 - typed line 14, Eq. 1.3 should read Eq. 1.1

Page 3 - line 4 at end of line, Eq. 1.5 should read Eq. 1.6

Page 8 - line 9, Ref. 3 should read Ref. 2

paragraph 2.2, line 1, Eq. 1.4 should read Eq. 1.9

Page 12 - typed line 4, Eq. 2.89 should read Eq. 2.88

Page 14 - Eq. (2.40), minus sign after = should be deleted

Page 16 - Eq. (2.45) should read:

$$\{Z_5\} = 2\,\mathrm{Real}\left[\{Z_5^{(5)}\}e^{i5\omega_f t_0} + \{Z_5^{(3)}\}e^{i3\omega_f t_0} + \{Z_5^{(1)}\}e^{i\omega_f t_0}\right]$$

Page 20 - Eq. (2.74) should read:

$$\delta_{3,0} = \Lambda_2^2 \lfloor U^T \rfloor [E][N][E]\{U\}$$

Page 24 - Eq. (2.93) should read:

$$\{P_3^{(3)}\} = [N]\{H_0\}A^3 = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} A^3$$

$$= p_1 \begin{Bmatrix} 1 \\ p \end{Bmatrix} A^3$$

Page 31 - Eq. (2.119), line 5 should read:

$$\delta_4 = \lfloor U^T \rfloor [S][N]\{H_1\}$$

Page 35 - typed line 6 should read:

Then the logarithm term goes to $\ln(-\gamma_R/\beta_R)$ as t_2 goes to infinity.* Thus, in

Page 43 - paragraph 3.2, line 1, Eq. A.1 should read Eq. 1.8
paragraph 3.2, line 2, Eqs. 3.2 and 3.3 should read Eqs. 3.3 and 3.4

Page 53 - Eq. (3.53) should read:

$$A = A_0 \, cn \left[\omega_u (t_1 + t_{1,0}), K_u \right]$$

typed line 2 should read:

The dependence of A_o, ω_u, and K_u upon β, γ, and $\mathcal{E}$ is obtained by combining

Page 54 - Eq. (3.58) should read:

$$A = A_0 \, dn \left[\omega_B (t_1 + t_{1,0}), K_B \right]$$

typed line 2 should read:

The dependence of A_o, ω_B and K_B upon β, γ and $\mathcal{E}$ is obtained by combining

Page 56 - second line of Eq. (3.66) should read:

$$+ \frac{\alpha_1}{\alpha} \left(\frac{\partial^3 A}{\partial t_1^3} + \beta_1 \frac{\partial A}{\partial t_1} + 3 \gamma_1 A^2 \frac{\partial A}{\partial t_1} \right) = 0$$

Page 59 - second line of footnote should read:

Eq. 3.123, but the terms of the type $\int Q \, dt_1$ are retained.

Page 73 - paragraph 4.1, line 14 should read:

avoided by a suitable choice of Λ_F. As a consequence, the solution does

Page 75 - Eq. (4.8) should read:

$$\{C_3\} = \begin{Bmatrix} C_{11} \bar{W}_{11}^3 + C_{12} \bar{W}_{11} \bar{W}_{21}^2 \\ C_{21} \bar{W}_{11}^2 \bar{W}_{21} + C_{22} \bar{W}_{21}^3 \end{Bmatrix}$$

Eq. (4.6) should read:

$$[\bar{G}] = \left\lceil \bar{g}_n \right\rfloor$$

Errata for ASRL TR 164-2 (Con't)
(March 1971)

Page 79 - Eq. (4.29), last line should read:

$$+i\omega_-\left(2\{U_-\}\frac{\partial A_-}{\partial t_1}+[\bar{G}]\{U_-\}A_-\right)e^{i\omega_- t_0}\Big]$$

Page 81 - Eq. (4.38) should read:

$$\Lambda_0^2=\frac{\bar{g}_1\bar{g}_2}{(\bar{g}_1+\bar{g}_2)^2}(\Omega_1^2-\Omega_2^2)^2$$

Page 82 - last typed line on page should read:

of Λ_0, can be rewritten as (see Eq. 4.32)

Page 83 - paragraph 4.5, line 1 should read:

Consider the vector $\{W_3\}$. Combining Eqs. 4.12, 4.39, and 4.44 yields

Page 84 - line 1 , Eq. (4.54) should read:

$$2i\omega\lfloor U^L\rfloor\{U\}\left(\frac{\partial B}{\partial t_1}+\frac{\partial A}{\partial t_2}\right)+i\omega\lfloor U^L\rfloor[\bar{G}]\{V\}A$$

Eq. (4.57) should read:

$$\bar{\beta}=\frac{1}{\alpha}\left(i\omega\lfloor U^L\rfloor[\bar{G}]\{V\}+\Lambda_2\lfloor U^L\rfloor[E]\{U\}\right)$$

Page 86 - Eq. (4.67) should read:

$$\{W_3\}=2\,Real\left[\left(c\{U\}+\{P_3^{(1)}\}\right)e^{i\omega t_0}+\{P_3^{(3)}\}e^{i3\omega t_0}\right]$$

Page 87 - last two typed lines should read:

yields, by taking into account that $\partial A/\partial t_1 = 0$ (Eq. 4.40) and $\partial B/\partial t_1 = 0$ (Eq. 4.55) and Eq. 4.70 , Page 88 - Eq. (4.74), line 6 should read:

$$+i3\omega[\bar{G}][P_3^{(2)}]e^{i3\omega t_0}+[\bar{G}]\{u\}\frac{\partial A}{\partial t_2}e^{i\omega t_0}$$

Page 90 - Eq. (4.81), line 3 should read:

$$-i\omega^3[\bar{G}][N][\bar{G}][N][\bar{G}]\,]\{u\}$$

last typed line should read:

Equation 4.79 is satisfied by

Page 95 - line 1 should read:

Summarizing, Eq. 4.84 can be solved by

Page 96 - Eq. (4.117), line 10 should read:

$$=2\,Re.\left[3(C_{21}u+C_{22}u^3)[A^2Be^{i3\tau}+(2AA^*B+A^2B^*)e^{i\tau}]\right]$$

Eq. (4.117) last line should read:

$$=2\,Re.\left[(C_{21}+3C_{22}u^2)v(A^3e^{i3\tau}+A^2A^*e^{i\tau})\right]$$

Page 97 - Eq. (4.120) should read:

$$A^2=-\frac{\bar{\beta}'-\bar{\beta}}{\bar{\gamma}'-\bar{\gamma}}$$

paragraph 4.9, typed line 3 should read:

where $\bar{\beta}$, $\bar{\gamma}$, $\bar{\beta}'$ and $\bar{\gamma}'$ are given by Eqs. 4.60, 4.81 and 4.82. Note that Eqs. 4.80

Page 98 - Eq. (4.123), first two lines should read:

$$\bar{\alpha}'=-4i\omega\bar{u}\bar{v}=\frac{\partial\bar{\alpha}}{\partial\bar{u}}\bar{v}$$

$$\bar{\alpha}'\bar{\beta}'=-\Lambda_2 2\bar{v}+i\omega\bar{g}_2\bar{v}^2=\frac{\partial\bar{\beta}}{\partial\bar{u}}\bar{v}$$

Page 98 - Eq. (4.124) should read:

$$\bar{\alpha}\bar{\alpha}'(\bar{\beta}'-\bar{\beta}) = 2i\omega\bar{v}(-2\Lambda_2 - i\omega\bar{g}_2\bar{v})(1+\bar{u}^2)$$

Eq. (4.125) should read:

$$\bar{\alpha}\bar{\alpha}'(\bar{\gamma}'-\bar{\gamma}) = 4i\omega\bar{u}\bar{v}\{3C_{11}+(1+2\bar{u}^2)C_{12}-(2+\bar{u}^2)C_{21} - 3C_{22}\bar{u}^2\}$$

Page 99 - typed lines 2 and 3 should read:

Section 2 of Ref. 2 and the coefficients $\bar{\beta}$ and $\bar{\gamma}$ (Eq. 4.60) and $\tilde{\beta}$ and $\tilde{\gamma}$ (Eq. 4.93) used here. Note that according to Eqs. 2.37 to 2.39 of Section 2, Ref. 2

typed line 8 should read:

with $\bar{u}$ given by Eq. 4.43 and $\bar{v}$ given by Eq. 4.47.

Page 100 - typed line 1 should read:

Comparing Eqs. 4.60, 4.123 and

Eq. (4.135), line 1 should read:

$$\beta = [(\bar{\alpha}\bar{\beta} + ig_2\bar{u}\bar{v}) + \varepsilon(\bar{\alpha}'\bar{\beta}' + ig_2\omega\bar{v}^2) + \cdots]$$

Eq. (4.135), lines 3 and 4 should read:

$$= \bar{\beta} + \frac{ig_2\bar{u}\bar{v}}{\bar{\alpha}} + \varepsilon\left[\frac{\bar{\alpha}'}{\bar{\alpha}}(\bar{\beta}'-\bar{\beta}) + \frac{ig_2\omega\bar{v}^2}{\bar{\alpha}}\right] + \cdots$$

$$= \left(\bar{\beta} + \frac{ig_2\bar{u}\bar{v}}{\bar{\alpha}}\right) + \varepsilon\left(\tilde{\beta} + i\frac{1}{\bar{\alpha}}g_2\omega\bar{v}^2\right) + \cdots$$

Page 102 - paragraph 5.3, line 1 should read:

In Section 3, the analysis is extended to include the behavior of the

Page 103 - typed line 7 should read:

Subsection 4.9); whereas β_I and γ_I are of order one. Thus setting

Eq. (5.5) should read:

$$|A| = \left[\frac{-\bar{\gamma}_R}{\bar{\beta}_R} + Ke^{\bar{\beta}_R\varepsilon^3 t}\right]^{-1/2}$$

Page 107 - Fig. 2; in middle of Fig. 2, arrow now labelled ε (adjacent to the angle θ) should be labelled ε^2.

Page 109 - Eq. (A.3), line 1 should read:

$$\Omega_n^2 = \Omega_{no}^2 \left(1 - \frac{N}{N_n}\right)$$

Page 110 - Eq. (A.4), line 1 should read:

$$\Omega_{no} = n^4 \pi^4$$

Page 114 - Eq. (A.31), line 1 should read:

$$\Lambda^2 < \Lambda_F^2 = \frac{g_1 g_2}{(g_1 + g_2)^2} \left(\Omega_2^2 - \Omega_1^2\right)^2$$

Page 118 - Eq. (A. 49) should read:

$$\frac{\Lambda_B^2}{\pi^8} = -\left(16 - 20\frac{N}{N_1} + 4\frac{N^2}{N_1^2}\right)$$

$$\frac{\Lambda_F^2}{\pi^8} = \frac{\theta}{(\theta+1)^2}\left(15 - \frac{3N}{N_1}\right)^2 + g^2\,\frac{\theta\left(1 - \frac{N}{N_1}\right) + \left(16 - 4\frac{N}{N_1}\right)}{\theta + 1}$$

typed line 14 should read:

The stability region in the λ,N plane is shown (for $g^2 = 0$ and $g^2 = .1$)

Page 119 - footnote, next to last equation line should read:

$$\left(p^2 + \frac{c}{a}\right)\left[p^2 + ap + \left(b - \frac{c}{a}\right)\right] = p^4 + ap^3 + bp^2 + cp + d$$

Page 121 - Eq. (A.63b) should read:

$$\Lambda^2 \geq \Lambda_B^2(N) \qquad \text{(buckling)}$$

Page 122 - typed line 9 should read:

pond to (see Eqs. A.44 and A.30)

Page 123 - typed line 3 should read:

where A and A_1 are two arbitrary constants and $\{U\}$ and $\{U_1\}$ are given by*

Page 126 - Figure A.1 - part of the legend on Fig. A.1 should read:

$$g^2 = \frac{g_1 g_2}{\Omega_{10}^2} = 0$$

Page 133 - last typed line should read:

Combining Eqs. B.10, B.11, and B.17 yields

Page 134 - Eq. (B.19), line 2 should read:

$$+2\frac{\partial A}{\partial t_4} + (\lambda_4 + i\frac{1}{4})A - i\mu A^2 A^*$$

Page 135 - Eq. (B.25), line 1 should read:

$$b_R = [(\frac{1}{K}\frac{\partial K}{\partial t_4} + \frac{\nu}{\mu^2} - \lambda_4)t_2 + C_1](\frac{1}{2} - 2\beta|A|^2)$$

Eq. (B.27), should read:

$$\frac{1}{K}\frac{\partial K}{\partial t_4} + \frac{\nu}{\mu^2} - \lambda_4 = 0$$

Page 136 - typed line 2 should read:

where k_o and ϕ_o are functions of t_6, t_8 . . . The value of λ_4 is still un-

Page 137 - Eq. (B.38) should read:

$$B = [-\frac{\nu}{\mu^2} K e^{-t_2}|A|^2 \ln|A| - i\frac{1}{4}\ln\{(4\mu)^{1/2}|A|\}]A$$

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16. Abstract

Nonlinear panel-flutter problems are analyzed by using the multiple time-scaling technique. The von Karman equations and the Galerkin method are used to derive a system of nonlinear ordinary differential equations. The three following analyses are presented in detail. First, the flutter solution is given up to the fifth order; this shows that the limit cycle predicted by the third-order analysis may disappear if a certain condition is satisfied. Second, the interaction of buckling and flutter is analyzed; the steady-state solution is given in terms of Jacobian elliptic functions. Third, the effect of small damping terms is analyzed. Limitations of the method are indicated and suggestions for future analysis are given.

A comparison with a generalization of the two-time-scaling technique by Cole and Kevorkian is also made. The results obtained with the two methods are in full agreement.

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FOREWORD

This report was prepared by the Aeroelastic and Structures Research Laboratory, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts under Grant No. NGR 22-009-387 for the Dynamic Loads Division, Langley Research Center, National Aeronautics and Space Administration, Hampton, Virginia 23365. Mr. Robert W. Hess of NASA-Langley served as technical advisor.

The advice and cooperation of Mr. Hess throughout this investigation are much appreciated. The authors wish especially to acknowledge the valuable advice of Professor John Dugundji of the MIT-ASRL on the theoretical aspects of this study; also, Professor Dugundji and Professor Emmett A. Witmer in the final review and editing of this report. The calculations were carried out in part at the MIT Information Processing Center.

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SECTION 1

INTRODUCTION

1.1 Formulation of the Problem

Panel flutter is the self-excited oscillation of a plate or shell when exposed to an airflow along its surface. Within the framework of the linear theory there is a critical value of the dynamic pressure, above which the motion of the plate is unstable and the amplitude of the oscillation grows exponentially in time. However, it is well known that the presence of (stabilizing) nonlinear effects induces a periodic motion whose amplitude does not depend upon the initial conditions.

This problem is analyzed in Refs. 1 and 2, where a detailed account of the methods used to solve the problem is also given. The analysis presented here is based upon the multiple time scaling technique* and is an extension of the analysis given in Ref. 1 (see also Ref. 2). Thus, for the sake of simplicity, the equations of the problem are given here without derivation. The details of the derivation can be found in Refs. 1 and 2. The analysis presented there is based upon the von Karman equations for a flat plate (see Eq. A.8 of Ref. 2).

$$D\left(1+\bar{g}_E\frac{\partial}{\partial t}\right)\nabla^4 W + \rho_m h\frac{\partial^2 W}{\partial t^2} = \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}$$

$$-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y} + N_x^{(a)}\frac{\partial^2 W}{\partial x^2} + N_y^{(a)}\frac{\partial^2 W}{\partial y^2} - (p-p_\infty)$$

$$\frac{1}{Eh}\nabla^4\phi = \left(\frac{\partial^2 W}{\partial x\partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} \tag{1.1}$$

The aerodynamic pressure is given by piston theory as (see Eq. A.9b of Ref.2)

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*See Subsection 1.3.

$$p - p_\infty = \frac{2q}{M}\left(\frac{\partial W}{\partial x} + \frac{1}{U}\frac{\partial W}{\partial t}\right) \tag{1.2}$$

By using the Galerkin method, the von Karman equations reduce to the following system of equations:

$$\ddot{W}_n + g_n \dot{W}_n + \Omega_n^2 W_n + \sum_{p=1}^{N} \lambda e_{np} W_p + \sum_{p,q,r=1}^{N} C_{npqr} W_p W_q W_r = 0 \tag{1.3}$$

where λ is the dynamic pressure parameter.

Explicit expressions for these coefficients in terms of the parameters of Eqs. 1.1 and 1.2 are given in Appendix A of Ref. 2 for an infinitely long simply-supported plate and for a rectangular simply-supported plate, and hence are not repeated here. It may be worth noting that the coefficients e_{np} which represent the aerodynamic forces are such that

$$e_{np} = -e_{pn} \tag{1.4}$$

and further that the coefficients C_{npqr} which represent the nonlinear effect of the membrane forces are such that

$$C_{npqr} = 0 \qquad \text{for } n + p + q + r = \text{odd} \tag{1.5}$$

Equation 1.5 (as well as the fact that there are no second-order nonlinear terms) is derived in Ref. 1 by using the fact that the plane of the panel is a plane of symmetry for the structure.

It should be emphasized that in Eq. 1.3, terms of higher order than w_i^3 have been neglected. If nonlinear terms up to the fifth order were included, Eq. 1.3 would become (see Ref. 3):

$$\ddot{W}_n + g_n \dot{W}_n + \Omega_n^2 W_n + \lambda \sum_{p=1}^{N} e_{np} W_p + \sum_{p,q,r=1}^{N} C_{npqr} W_p W_q W_r$$
$$+ \sum_{p,q,r,s,t=1}^{N} d_{npqrst} W_p W_q W_r W_s W_t = 0 \tag{1.6}$$

It should be noted again that by using the fact that the plane of the panel is a plane of symmetry for the structure yields the result that:

$$d_{npqrst} = 0 \qquad \text{for } n + p + q + r + s + t = \text{odd} \tag{1.7}$$

as well as that there are no fourth-order nonlinear terms (see Ref. 1).

For the sake of simplicity, only the two-mode case is considered in detail here (although some results are generalized to the N-mode case). As shown in Ref. 1, taking into account Eqs. 1.4, 1.5, and 1.7, Eqs. 1.3 and 1.5 for the two-mode case reduce, respectively, to

$$\ddot{W}_1 + g_1 \dot{W}_1 + \Omega_1^2 W_1 - \Lambda W_2 + (C_{11} W_1^2 + C_{12} W_2^2) W_1 = 0$$

$$\ddot{W}_2 + g_2 \dot{W}_2 + \Omega_2^2 W_2 + \Lambda W_1 + (C_{21} W_1^2 + C_{22} W_2^2) W_2 = 0 \tag{1.8}$$

and

$$\ddot{W}_1 + g_1 \dot{W}_1 + \Omega_1^2 W_1 - \Lambda W_2 + (C_{11} W_1^2 + C_{12} W_2^2) W_1 + (d_{11} W_1^4 + d_{12} W_1^2 W_2^2 + d_{13} W_2^4) W_1 = 0$$

$$\ddot{W}_2 + g_2 \dot{W}_2 + \Omega_2^2 W_2 + \Lambda W_1 + (C_{21} W_1^2 + C_{22} W_2^2) W_2 + (d_{21} W_1^4 + d_{22} W_1^2 W_2^2 + d_{23} W_2^4) W_2 = 0 \tag{1.9}$$

In Eqs. 1.8 and 1.9, Λ is given by

$$\Lambda = \lambda e_{21} \tag{1.10}$$

and the coefficients c_{ik} are given by

$$C_{11} = C_{1111} \qquad C_{12} = C_{1122} + C_{1212} + C_{1221}$$

$$C_{21} = C_{2211} + C_{2121} + C_{2112} \qquad C_{22} = C_{2222} \tag{1.11}$$

and similar expressions hold for the coefficients d_{ik}.

1.2 Outline of the Analysis

The analysis presented here is an extension of those described in Refs. 1 and 2. The analysis described in Ref. 2 is limited in three respects:

(a) It is limited to third-order nonlinear terms.

(b) It is assumed that the flutter frequency is not "too small" (not of order ε).

(c) It is assumed implicitly that the damping terms are not "too small" (not of order ε).

These limitations are eliminated in Sections 2, 3, and 4 of this report.

In Section 2, Eq. 1.9 (which includes the fifth-order nonlinear terms) is studied in detail. It is found that the effect of the fifth-order nonlinear terms can reverse the trend obtained with the third-order nonlinear analysis. A detailed description of this effect is given in Subsection 2.1 where a summary of the essential results contained in Section 2 is also given.

Next, the interaction of buckling and flutter is considered. First, the linear problem is examined in detail. For clarity and convenience, the analysis of the linear problem is given in Appendix A. The analysis of the linear problem shows that at the intersection of the flutter-stability boundary with the buckling-stability boundary, the flutter frequency is equal to zero. Thus, the analysis given in Ref. 1 is not valid in the neighborhood of this intersection point. The solution for Eq. 1.8 is obtained in terms of Jacobian elliptic functions cn(u) and dn(u). A detailed summary of the results obtained in Section 3 is given in Subsection 3.1.

In Section 4, the analysis given in Ref. 1 is repeated by assuming that the coefficients g_n of Eq. 1.8 are of order ε. The results are in very good agreement with those obtained in Ref. 1. A detailed summary of the results obtained in Section 4 is given in Subsection 4.1.

It should be noted that the analyses given in Sections 3 and 4 deal with an interesting mathematical problem. The analyses can predict the steady-state solution, but cannot predict the transient response. More specifically, during the transient response, it is possible to predict the variation of the amplitude, but the variation of frequency cannot be obtained. An interpretation of this phenomenon, as well as a suggestion for future work, are given in Section 5.

Finally, as mentioned in Subsection 2.1, in the analysis of the effect of the fifth-order nonlinear terms, a condition based upon physical reasoning is used in order to determine the value of one of the parameters of the solution. Since no mathematical motivation is given there, it seems appropriate to apply the same method of solution to a case for which the exact mathematical solution is available. Accordingly, a problem of this kind is analyzed in Appendix B where the results obtained are found to be in complete agreement with the exact solution of the problem.

1.3 The Multiple Time-Scaling Method: Secular and Subsecular Terms

The reader is assumed to be familiar with the multiple-time-scaling technique and, in particular, with the concept of secular terms and the condition for no-secular-terms (i.e., the exclusion of secular terms); a detailed introduction to the method, however, is given in Section 2 of Ref. 2. For convenience, the main concepts are given in the following. The details can be found in Ref. 2.

By using a perturbation method, one may encounter terms of the type $[t \sin \omega t]$ or of a similar type, whereas from a physical point of view one does not expect a solution of this type, since a bounded solution is expected. These terms (which show the "singular" nature of the problem) are called secular terms. One of the singular perturbation techniques used to solve this type of problem is the multiple-time-scaling technique. In this technique, multiple time scales, $t_o = t$, $t_1 = \varepsilon t$, $t_2 = \varepsilon^2 t$, etc., are introduced in order to increase the versatility of the system. In order to avoid secular terms, one explicitly introduces a "no-secular-term condition" (see Eq. 1.13) which generally yields the dependence of the solution upon the "slow" time scales $(t_1, t_2, \ldots)$.

It should be noted that in this report, secular terms are encountered in the solution of the no-secular-terms condition: for the sake of simplicity, these terms will be called subsecular terms.

Finally the condition for no-secular terms is stated. It should be noted that if L is a linear operator, the equation $L(P) = Z$ has a solution only if Z is orthogonal to U^L where U^L is the solution of the homogeneous equation in terms of the adjoint operator. In particular, in matrix notation, if $\{U^L\}$ is a solution of the equation $\lfloor U^L \rfloor$ $[M] = 0$, the equation

$$[M]\{P\} = \{Z\} \tag{1.12}$$

has a solution only if

$$\lfloor U^L \rfloor \{Z\} = 0 \tag{1.13}$$

Equation 1.13 is the <u>condition for no-secular-terms</u> to be present in the solution.

SECTION 2

THE EFFECT OF THE FIFTH-ORDER NONLINEAR TERMS

2.1 Introduction

As mentioned in Section 1, the analysis of nonlinear panel flutter given in Refs. 1 and 2 is limited to including membrane-force terms only up through the third order nonlinear terms. As is well known, the inclusion of the fifth-order nonlinear terms may reverse completely the trend obtained with the third-order nonlinear analysis; for instance, in the case discussed in Ref. 3, the third-order nonlinear analysis reveals the existence of an unstable limit cycle, whereas the fifth-order nonlinear terms are stabilizing with the result that for higher values of the amplitude of vibration, the unstable limit cycle disappears and is replaced by a stable limit cycle. Thus, in order to understand better the mechanism of the effect of the fifth-order nonlinear terms, the analysis given in Ref. 2 is extended up to the fifth order in this report.

It will be seen in Subsection 2.2 that by using the expansion for $\{W\}$ given by Eq. 2.9, the time scales $t_n = \varepsilon^n t$ given by Eq. 2.11, and then by separating terms of the same order, Eq. 2.1 yields the set of recurrence relations given by Eqs. 2.17, 2.18, and 2.19. The first two equations yield exactly the same results as obtained with the third-order analysis given in Ref. 2. These results are summarized in Subsection 2.3. The remainder of the section is devoted to the solution of Eq. 2.19. Before discussing the results of this solution, it should be emphasized that a parameter Λ_4 is introduced in the expansion of Eq. 2.10. This provides greater versatility in the equation, and the value of this parameter can be used to satisfy a further condition (discussed later).

Equation 2.19 is discussed in Subsection 2.4; however, the results are summarized here. In order to avoid secular terms in the solution of Eq. 2.19, the condition given by Eq. 2.54 must be satisfied. This corresponds to a differential equation for B as a function of t_2 (B is the arbitrary coefficient of the solution for $\{W_3\}$, see Eq. 2.36). The solution of this equation (written to avoid secular terms) will contain subsecular terms unless Eqs. 2.71 and 2.72 are satisfied (the details of this analysis are given in Subsection 2.7).

These two equations combined with Eqs. 2.31 to 2.33 give the dependence of A on t_4 (Eq. 2.77).

The solution obtained thus far, still contains an arbitrary parameter Λ_4. As mentioned above, this parameter is used to satisfy a further condition which can be described as follows. Consider the curve which gives the value of the limit-cycle amplitude in terms of the dynamic pressure parameter Λ. For "stabilizing" third-order terms and "destabilizing" fifth-order terms this curve is qualitatively depicted in Fig. 1. It is well known (see, for instance, Ref. 3) that the lower branch is stable whereas the upper branch is unstable. Thus, the following condition must be satisfied: the value ε_{knee} of ε corresponding to the maximum value of Λ on the "fifth order" curve of Fig. 1 is equal to the value (ε_{cr}) of ε for which the nature of the limit cycle changes (from stable to unstable). The condition $\varepsilon_{cr} = \varepsilon_{knee}$ defines the value of Λ_4 (see Eq. 2.84). The details of the analyses are given in Subsection 2.5.

It should be noted that this condition is not based upon mathematical reasoning, but only on physical intuition. Thus, it is useful to verify the correctness of this intuition by applying this analysis to a problem for which the exact solution is known. Hence, a one-degree-of-freedom problem for which the exact solution is known is discussed in Appendix B. The results obtained by using the physical assumption outlined above are in perfect agreement with the exact solution.

2.2 General Formulation

Consider Eq. 1.4. In matrix notation this equation can be written as

$$\frac{d^2}{dt^2}\{W\} + [G]\frac{d}{dt}\{W\} + [\Omega^2]\{W\} + \Lambda[E]\{W\} + \{C\} + \{D\} = 0 \tag{2.1}$$

where

$$\{W\} = \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} \tag{2.2}$$

$$[G] = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \tag{2.3}$$

$$[\Omega^2] = \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \tag{2.4}$$

$$[E] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.5}$$

$$\{C\} = \begin{Bmatrix} (C_{11} W_1^2 + C_{12} W_2^2) W_1 \\ (C_{21} W_1^2 + C_{22} W_2^2) W_2 \end{Bmatrix} \tag{2.6}$$

$$\{D\} = \begin{Bmatrix} (d_{11} W_1^4 + d_{12} W_1^2 W_2^2 + d_{13} W_2^4) W_1 \\ (d_{21} W_1^4 + d_{22} W_1^2 W_2^2 + d_{23} W_2^4) W_2 \end{Bmatrix} \tag{2.7}$$

Equation 2.1 can be solved by using the multiple-time-scaling technique. The solution was obtained in Subsection 2.3 of Ref. 2 up to terms of order ε^3. In this section, the solution up to terms of order ε^5 is obtained. By analogy to the procedure used in Ref. 2, set

$$W_k = \varepsilon w_{k1} + \varepsilon^3 w_{k3} + \varepsilon^5 w_{k5} + \cdots \tag{2.8}$$

or, in vector notation

$$\{W\} = \varepsilon \{W_1\} + \varepsilon^3 \{W_3\} + \varepsilon^5 \{W_5\} + \cdots \tag{2.9}$$

and*

$$\Lambda = \Lambda_F + \varepsilon^2 \Lambda_2 + \varepsilon^4 \Lambda_4 + \cdots \tag{2.10}$$

---

*As is shown in Ref. 2, Λ_2 assumes the value ± 1 (see Subsection 2.7).

and introduce the time scales

$$
\begin{aligned}
t_0 &= t \\
t_2 &= \varepsilon^2 t \\
t_4 &= \varepsilon^4 t \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{aligned}
\tag{2.11}
$$

Equation 2.11 implies that

$$
\frac{d}{dt} = \left(\frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^4 \frac{\partial}{\partial t_4} + \cdots \right) \tag{2.12}
$$

Combining Eqs. 2.1, 2.9, 2.10, and 2.12 yields

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^4 \frac{\partial}{\partial t_4} + \cdots \right)^2 \left(\varepsilon \{W_1\} + \varepsilon^3 \{W_3\} + \varepsilon^5 \{W_5\} + \cdots \right) \\
&+ [G] \left(\frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^4 \frac{\partial}{\partial t^4} + \cdots \right) \left(\varepsilon \{W_1\} + \varepsilon^3 \{W_3\} + \varepsilon^5 \{W_5\} + \cdots \right) \\
&+ [\Omega^2] \left(\varepsilon \{W_1\} + \varepsilon^3 \{W_3\} + \varepsilon^5 \{W_5\} + \cdots \right) \\
&+ \left(\Lambda_F + \varepsilon^2 \Lambda_2 + \varepsilon^4 \Lambda_4 + \cdots \right) [E] \left(\varepsilon \{W_1\} + \varepsilon^3 \{W_3\} + \varepsilon^5 \{W_5\} + \cdots \right) \\
&+ \varepsilon^3 \{C_3\} + \varepsilon^5 \{C_5\} + \cdots + \varepsilon^5 \{D_5\} = 0
\end{aligned}
\tag{2.13}
$$

where

$$
\{C_3\} = \left\{ \begin{matrix} C_{11} W_{11}^3 + C_{12} W_{11} W_{21}^2 \\ C_{21} W_{11}^2 W_{21} + C_{22} W_{21}^3 \end{matrix} \right\} \tag{2.14}
$$

$$
\{C_5\} = \left\{ \begin{matrix} 3 C_{11} W_{11}^2 W_{13} + (W_{13} W_{21}^2 + 2 W_{11} W_{21} W_{23}) C_{12} \\ (2 W_{11} W_{13} W_{21} + W_{11}^2 W_{23}) C_{21} + 3 W_{21}^2 W_{23} C_{22} \end{matrix} \right\} \tag{2.15}
$$

and

$$\{D_5\} = \begin{Bmatrix} W_{11}^5 d_{11} + W_{11}^3 W_{21}^2 d_{12} + W_{11} W_{21}^4 d_{13} \\ W_{11}^4 W_{21} d_{21} + W_{11}^2 W_{21}^3 d_{22} + W_{21}^5 d_{23} \end{Bmatrix} \tag{2.16}$$

Equating terms of the same order of magnitude (terms multiplying like power of ε) yields the following set of recurrent systems

Order ε

$$\frac{\partial^2}{\partial t_0^2}\{W_1\} + [G]\frac{\partial}{\partial t_0}\{W_1\} + [\Omega^2]\{W_1\} + \Lambda_F[E]\{W_1\} = 0 \tag{2.17}$$

Order ε^3

$$\frac{\partial^2}{\partial t_0^2}\{W_3\} + [G]\frac{\partial}{\partial t_0}\{W_3\} + [\Omega^2]\{W_3\} + \Lambda_F[E]\{W_3\} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\{W_1\}$$
$$+ [G]\frac{\partial}{\partial t_2}\{W_1\} + \Lambda_2[E]\{W_1\} + \{C_3\} = 0 \tag{2.18}$$

Order ε^5

$$\frac{\partial^2}{\partial t_0^2}\{W_5\} + [G]\frac{\partial}{\partial t_0}\{W_5\} + [\Omega^2]\{W_5\} + \Lambda_F[E]\{W_5\}$$
$$+ 2\frac{\partial^2}{\partial t_0 \partial t_2}\{W_3\} + \left(2\frac{\partial^2}{\partial t_0 \partial t_4} + \frac{\partial^2}{\partial t_2^2}\right)\{W_1\} + [G]\left(\frac{\partial}{\partial t_2}\{W_3\} + \frac{\partial}{\partial t_4}\{W_1\}\right)$$
$$+ \Lambda_2[E]\{W_3\} + \Lambda_4[E]\{W_1\} + \{C_5\} + \{D_5\} = 0 \tag{2.19}$$

2.3 The Solution with an Error of Order ε^5

The solution of Eq. 2.1 with an error of order ε^5, implies the analysis of Eqs. 2.17 and 2.18. This analysis is described in detail in Subsection 2.2 of Ref. 2. For the sake of convenience the analysis is repeated briefly here, using slightly different notations which are suitable to study the effect of the fifth-order nonlinear terms, described in Subsection 2.4. The solution of Eq. 2.17 is discussed in Subsection A.7 and is given by Eq. A.58.

$$\{W_1\} = A\{U\}e^{i\omega_F t_0} + A^*\{U^*\}e^{-i\omega_F t_0}$$
$$= 2\,\mathrm{Real}\left(A\{U\}e^{i\omega_F t_0}\right) \tag{2.20}$$

where $\{U\}$ is given by Eq. A.59 and ω_F is given by Eq. A.51. In Eq. 2.20, the parameter A is a complex function of t_2, t_4, . . . Combining Eqs. 2.18 and

2.20 yields

$$\frac{\partial^2}{\partial t_o^2}\{W_3\}+[G]\frac{\partial}{\partial t_o}\{W_3\}+[\Omega^2]\{W_3\}+\Lambda_F[E]\{W_3\}+\{Z_3\}=0 \tag{2.21}$$

with

$$\{Z_3\}=2\,Real\left[(2i\omega_F[I]+[G])\{U\}\frac{\partial A}{\partial t_2}e^{i\omega_F t_o}+\Lambda_2[E]\{U\}Ae^{i\omega_F t_o}\right.$$
$$\left.+\{C_3\}\right]=2\,Real\left[Z_3^{(3)}e^{i3\omega_F t_o}+Z_3^{(1)}e^{i\omega_F t_o}\right] \tag{2.22}$$

where

$$[Z_3^{(3)}]=\{H_o\}A^3 \tag{2.23}$$

$$[Z_3^{(1)}]=(2i\omega_F[I]+[G])\{U\}\frac{\partial A}{\partial t_2}+\Lambda_2[E]\{U\}A$$
$$+\{H_1\}A^2A^* \tag{2.24}$$

with $\{H_o\}$ and $\{H_1\}$ given by Eqs. 2.87 and 2.89.

The solution for this system is discussed in Subsection 2.2 of Ref. 2. In order to avoid secular terms, $\{Z_3^{(1)}\}$ must be such that (see Eq. 1.13)

$$\lfloor U^L\rfloor\{Z_3^{(1)}\}=0 \tag{2.25}$$

This condition yields a differential equation for A

$$\frac{\partial A}{\partial t_2}+\beta A+\gamma A^2A^*=0 \tag{2.26}$$

where β and γ are given by Eqs. 2.126 and 2.127 of Ref. 2 or

$$\beta=\frac{1}{\alpha}\Lambda_2\lfloor U^L\rfloor[E]\{U\} \tag{2.27}$$

$$\gamma = \frac{1}{\alpha} \lfloor U^L \rfloor \{H_1\} \tag{2.28}$$

with

$$\alpha = \lfloor U^L \rfloor \left(2i\omega [I] + [G] \right) \{U\} \tag{2.29}$$

The vector $\lfloor U^L \rfloor$ is defined in Subsection 1.3 and is given by (see Eq. A.62)

$$\lfloor U^L \rfloor = \lfloor 1 \, , -u \rfloor \tag{2.30}$$

with u given by Eq. A.60.

The solution for Eq. 2.26 is given by Eq. 2.72 in Subsection 2.2 of Ref. 2 as

$$A = |A| \, e^{i\varphi} \tag{2.31}$$

with

$$|A| = \left(-\frac{\gamma_R}{\beta_R} + K \, e^{2\beta_R t_2} \right)^{-1/2} \tag{2.32}$$

and

$$\varphi = -\left(\beta_I - \frac{\beta_R}{\gamma_R} \gamma_I \right) t_2 + \frac{\gamma_I}{\gamma_R} \ln |A| + \varphi_0 \tag{2.33}$$

where

$$\beta = \beta_R + i \beta_I \tag{2.34}$$

$$\gamma = \gamma_R + i\gamma_I \tag{2.35}$$

Finally, if A is given by Eq. 2.31, the condition for no secular terms (Eq. 2.25) is satisfied and the solution for Eq. 2.21 is given by

$$\{W_3\} = 2\,\mathrm{Real}\left[\left(B\{U\} + \{P_3^{(1)}\}\right)e^{i\omega_F t_0} + \{P_3^{(3)}\}e^{i3\omega_F t_0}\right] \tag{2.36}$$

where

$$\{P_3^{(1)}\} = [\,N\,]\{Z_3^{(1)}\} \tag{2.37}$$

$$\{P_3^{(3)}\} = [\,L^{(3)}\,]\{Z_3^{(3)}\} \tag{2.38}$$

with

$$[N] = -\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{-\omega_F^2 + i g_2 \omega_F + \Omega_2^2} \end{bmatrix}$$

$$[\,L^{(3)}\,] = -[M^{(3)}]^{-1} \tag{2.39}$$

where

$$[M^{(3)}] = -\begin{bmatrix} -9\omega_F^2 + i3g_1\omega_F + \Omega_1^2 & -\Lambda_F \\ \Lambda_F & -9\omega_F^2 + i3g_2\omega_F + \Omega_2^2 \end{bmatrix} \tag{2.40}$$

It is convenient to repeat here the differential equations for $|A|$ and ϕ (obtained in Subsection 2.2, Ref. 2):

$$\frac{\partial |A|}{\partial t_2} + \beta_R |A| + \gamma_R |A|^3 = 0 \tag{2.41}$$

$$\frac{\partial \varphi}{\partial t_2} + \beta_I + \gamma_I |A|^2 = 0 \tag{2.42}$$

2.4 The Solution with an Error of Order ε^7

In this subsection, the expression for $\{W_5\}$ is obtained by solving Eq. 2.19. Combining Eqs. 2.19, 2.20, and 2.36 yields

$$\frac{\partial^2}{\partial t_0^2}\{W_5\} + [G]\frac{\partial}{\partial t_0}\{W_5\} + [\Omega^2]\{W_5\} + \Lambda_F[E]\{W_5\} + \{Z_5\} = 0 \tag{2.43}$$

where

$$\begin{aligned}
\{Z_5\} = 2\,\mathrm{Re.}\Bigg[&\left([I]\,2\frac{\partial^2}{\partial t_0 \partial t_2} + [G]\frac{\partial}{\partial t_2} + \Lambda_2[E]\right)\{W_3\} \\
&+ \left([I]\left(2\frac{\partial^2}{\partial t_0\partial t_4} + \frac{\partial^2}{\partial t_2^2}\right) + [G]\frac{\partial}{\partial t_4} + \Lambda_4[E]\right)\{W_1\} + \{C_5\} + \{D_5\}\Bigg] \\
= 2\,\mathrm{Re.}\Bigg[&\left([I]\,2\frac{\partial^2}{\partial t_0\partial t_2} + [G]\frac{\partial}{\partial t_2} + \Lambda_2[E]\right)\left(B\{u\}e^{i\omega_F t_0} + \{P_3^{(1)}\}e^{i\omega_F t_0}\right. \\
&\left. + \{P_3^{(3)}\}e^{i3\omega_F t_0}\right) + \left([I]\left(2\frac{\partial^2}{\partial t_0\partial t_4} + \frac{\partial^2}{\partial t_2^2}\right) + [G]\frac{\partial}{\partial t_4} + \Lambda_4[E]\right) \\
&\times \left(A\{u\}e^{i\omega_F t_0}\right) + \left(\{k_5^{(5)}\}e^{i5\omega_F t_0} + \{k_5^{(3)}\}e^{i3\omega_F t_0}\right. \\
&\left. + \{k_5^{(1)}\}e^{i\omega_F t_0}\right)\Bigg]
\end{aligned} \tag{2.44}$$

where $\{k_5^{(5)}\}$, $\{k_5^{(3)}\}$, and $\{k_5^{(1)}\}$are given by Eqs. 2.108, 2.109, and 2.110, respectively. Equation 2.44 can be rewritten as

$$\{Z_5\} = 2\,Real\left[Z_5^{(5)} e^{i5\omega_F t_0} + Z_5^{(3)} e^{i3\omega_F t_0} + Z_5^{(1)} e^{i\omega_F t_0} \right] \tag{2.45}$$

with

$$\{Z_5^{(5)}\} = \{k_5^{(5)}\} \tag{2.46}$$

$$\{Z_5^{(3)}\} = \left([I]\,2i\omega_F \frac{\partial}{\partial t_2} + [G]\frac{\partial}{\partial t_2} + \Lambda_2 [E] \right)\{P_3^{(3)}\} + \{k_5^{(3)}\} \tag{2.47}$$

$$\begin{aligned}\{Z_5^{(1)}\} = {} & \left([I]\,2i\omega_F \frac{\partial}{\partial t_2} + \Lambda_2 [E] + [G]\frac{\partial}{\partial t_2} \right)\left(B\{U\} + \{P_3^{(1)}\} \right) + \\ & + \left([I]\,2i\omega_F \frac{\partial}{\partial t_4} + [I]\frac{\partial^2}{\partial t_2^2} + [G]\frac{\partial}{\partial t_4} + \Lambda_4 [E] \right) A\{U\} + \{k_5^{(1)}\}\end{aligned} \tag{2.48}$$

In order to avoid secular terms, the following condition must be satisfied.

$$\lfloor U^L \rfloor \{Z_5^{(1)}\} = 0 \tag{2.49}$$

Equation 2.49 yields a differential equation for the function B which is discussed in Subsection 2.5. Once Eq. 2.49 is satisfied, the solution is given by

$$\{W_3\} = 2\,Real\left[\left(C\{U\} + \{P_5^{(1)}\} \right) e^{i\omega_F t_0} + \{P_5^{(3)}\} e^{i3\omega_F t_0} + \{P_5^{(5)}\} e^{i5\omega_F t_0} \right] \tag{2.50}$$

with

$$\{P_5^{(1)}\} = -\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{-\omega_F^2 + i\omega_F g_2 + \Omega_i^2} \end{bmatrix} \{Z_5^{(1)}\} = [N]\{Z_5^{(1)}\} \tag{2.51}$$

$$\{P_5^{(3)}\} = -\begin{bmatrix} -9\omega_F^2 + i3\omega_F g_1 + \Omega_1^2 & -\Lambda_F \\ \Lambda_F & -9\omega_F^2 + i3\omega_F g_2 + \Omega_2^2 \end{bmatrix}^{-1} \{Z_5^{(3)}\} = [L^{(3)}]\{Z_5^{(3)}\} \tag{2.52}$$

$$\{P_5^{(5)}\} = -\begin{bmatrix} -25\omega_F^2 + i5\omega_F g_1 + \Omega_1^2 & -\Lambda_F \\ \Lambda_F & -25\omega_F^2 + i5\omega_F g_2 + \Omega_2^2 \end{bmatrix}^{-1} \{Z_5^{(5)}\} = [L^{(5)}]\{Z_5^{(5)}\} \tag{2.53}$$

2.5 The Functions $B(t_2)$ and $A(t_2, t_4)$

In the preceding section, it was shown that in order to avoid secular terms in the solution for $\{W_5\}$ the following condition

$$\lfloor U^L \rfloor \{Z_5^{(1)}\} = 0 \tag{2.54}$$

must be satisfied. In Eq. 2.54, $\lfloor U^L \rfloor$ is given by Eq. 2.30 and $\{Z_5^{(1)}\}$ is given by Eq. 2.48, from which an explicit expression (Eq. 2.116) is derived in Subsection 2.7.1. This explicit expression is then used, in Subsection 2.7.2 to derive, from Eq. 2.54, a differential equation for B, given by Eq. 2.117:

$$\frac{\partial B}{\partial t_2} + \beta B + \gamma(A^2 B^* + 2AA^* B) + \frac{\delta}{\alpha} = 0 \tag{2.55}$$

with δ given by Eq. 2.123

$$\delta = \left[\delta^{(0)} + \delta^{(1)} AA^* + \delta^{(2)}(AA^*)^2 + \frac{\partial}{\partial t_4}\ln A\right]\alpha A \tag{2.56}$$

with $\delta^{(0)}$, $\delta^{(1)}$, and $\delta^{(2)}$ given by Eq. 2.124.

In order to solve Eq. 2.55, let

$$B = bA \tag{2.57}$$

This yields

$$\frac{\partial b}{\partial t_2} A + b\frac{\partial A}{\partial t_2} + \beta bA + \gamma(A^2A^*b^* + 2A^2A^*b) + \frac{\delta}{\alpha} = 0 \tag{2.58}$$

or, by using Eq. 2.26

$$\frac{\partial b}{\partial t_2} + \gamma AA^*(b + b^*) + \frac{\delta}{\alpha A} = 0 \tag{2.59}$$

Setting

$$b = b_R + i\, b_I \tag{2.60}$$

and separating real and imaginary parts yields

$$\frac{\partial b_R}{\partial t_2} + 2\gamma_R |A|^2 b_R + \left[\frac{\delta}{\alpha A}\right]_R = 0 \tag{2.61}$$

$$\frac{\partial b_I}{\partial t_2} + 2\gamma_I |A|^2 b_R + \left[\frac{\delta}{\alpha A}\right]_I = 0 \tag{2.62}$$

Equations 2.61 and 2.62 are discussed in Subsection 2.7. The solutions for these equations are given by (see Eqs. 2.152 and 2.153)

$$b_R = b_R^{(0)} + b_R^{(1)}|A|^2 + b_R^{(2)} \ln|A| + b_R^{(3)} |A|^2 \ln|A| \tag{2.63}$$

$$b_I = b_I^{(o)} + b_I^{(1)}|A|^2 + b_I^{(2)} \ln|A| + b_I^{(3)}|A|^2 \ln|A| \qquad (2.64)$$

where $b_R^{(o)}$ and $b_I^{(o)}$ are constants of integration, whereas $b_R^{(i)}$ and $b_I^{(i)}$ are given by Eqs. 2.154 and 2.155:

$$b_R^{(1)} = -\frac{\gamma_R}{2\beta_R^2}\left[\delta_R^{(o)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] + b_R^{(o)}\frac{\gamma_R}{\beta_R} \qquad (2.65)$$

$$b_R^{(2)} = \frac{\beta_R}{\gamma_R^2}\,\delta_R^{(2)} \qquad (2.66)$$

$$b_R^{(3)} = \frac{1}{\gamma_R}\,\delta_R^{(2)} \qquad (2.67)$$

and

$$b_I^{(1)} = \frac{\gamma_I}{\gamma_R} b_R^{(1)} + \frac{1}{2\gamma_R}\left(\delta_I^{(2)} - \frac{\gamma_I}{\gamma_R}\,\delta_R^{(2)}\right) \qquad (2.68)$$

$$b_I^{(2)} = \frac{1}{\gamma_R}\left(\delta_I^{(1)} - \frac{\beta_R}{\gamma_R}\delta_I^{(2)}\right) - \frac{\gamma_I}{\gamma_R^2}\left[\delta_R^{(1)} - 2\frac{\beta_R}{\gamma_R}\delta_R^{(2)}\right] \qquad (2.69)$$

$$b_I^{(3)} = \frac{\gamma_I}{\gamma_R}\,b_R^{(3)} \qquad (2.70)$$

It should be noted that in the process of solving Eqs. 2.61 and 2.62, "subsecular terms"* were encountered. As shown in Subsection 2.7.3, in order to avoid subsecular terms, the following expressions for $k(t_4)$ and $\phi(t_4)$ are obtained

*"Subsecular terms" are "secondary" secular terms obtained in the solution for the equation which must be satisfied in order to avoid the "principal" secular terms (see Subsection 1.3).

(see Eqs. 2.136 and 2.150)

$$K = K_o e^{2[\delta_R^{(o)} - \delta_R^{(2)}(\frac{\beta_R}{\gamma_R})^2] t_4} \tag{2.71}$$

$$\varphi_o = \Phi_1 t_4 + \Phi_o \tag{2.72}$$

where Φ_1 is given by Eq. 2.151, whereas k_o and Φ_o are functions of t_6, t_8, . . .

It should be noted that thus far, no condition on Λ_4 has been obtained. Thus, there is still one degree of arbitrariness in the solution. In other words, from a strict mathematical viewpoint, the constant Λ_4 can be chosen in an arbitrary way. In the following, a convenient choice of Λ_4 (based upon a physical point of view) is described. First, note that in Eq. 2.119, Λ_4 appears only in the definition of δ_3 which can be rewritten as

$$\delta_3 = \delta_{3,0} + \frac{\alpha\beta\Lambda_4}{\Lambda_2} \tag{2.73}$$

where

$$\delta_{3,0} = \Lambda_2^2 \lfloor U^L \rfloor [E][L^{(3)}][E]\{U\} \tag{2.74}$$

is the value of δ_3 for $\Lambda_4 = 0$.

Next, note that in Eq. 2.124, δ_3 appears only in the definition of $\delta^{(o)}$ which can thus be rewritten as

$$\begin{aligned} \delta^{(o)} &= \frac{1}{\alpha}\left(\delta_{3,0} + \frac{\alpha\beta\Lambda_4}{\Lambda_2} - \delta_2\beta + \delta_1\beta^2\right) \\ &= \delta_o^{(o)} + \frac{\beta\Lambda_4}{\Lambda_2} \end{aligned} \tag{2.75}$$

where

$$\delta_0^{(0)} = \frac{1}{\alpha}\left(\delta_{3,0} - \delta_2 \beta + \delta_1 \beta^2 \right) \tag{2.76}$$

is the value of $\delta^{(o)}$ for $\Lambda_4 = 0$. Next, note that combining Eq. 2.32 and Eq. 2.71

$$|A| = \left(-\frac{\gamma_R}{\beta_R} + K\, e^{2\beta_R t_2}\right)^{-1/2}$$

$$= \left[-\frac{\gamma_R}{\beta_R} + K_0\, e^{\left(2\beta_R t_2 + 2\left[\delta_R^{(0)} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] t_4\right)}\right]^{-1/2} \tag{2.77}$$

with $t_2 = \varepsilon^2 t$ and $t_4 = \varepsilon^4 t$. As mentioned in Subsection 2.2 of Ref. 2, the nature of the limit cycle (stable or unstable) depends upon the sign of the exponent

$$\left\{ \beta_R + \varepsilon^2 \left[\delta_R^{(0)} - \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R}\right)^2 \right] \right\} 2\varepsilon^2 t \tag{2.78}$$

Thus, there is a critical value ε^2_{cr}

$$\varepsilon_{cr} = \sqrt{\frac{-\beta_R}{\delta_R^{(0)} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2}} \tag{2.79}$$

such that if

$$\varepsilon > \varepsilon_{cr} \tag{2.80}$$

the nature of the limit-cycle changes.

The physical assumption is that this value of ε is equal to the value ε_{knee} at which Λ assumes its maximum value (knee of the curve which gives the amplitude as a function of Λ); that is (see Eq. 2.10)

$$\varepsilon_{knee} = \sqrt{-\frac{\Lambda_2}{2\Lambda_4}} \tag{2.81}$$

Equating Eqs. 2.79 and 2.81 yields

$$\delta_R^{(0)} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2 = \frac{2\Lambda_4\beta_R}{\Lambda_2} \tag{2.82}$$

or, by using Eq. 2.75

$$\delta_{oR}^{(o)} + \beta_R\frac{\Lambda_4}{\Lambda_2} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2 = \frac{2\Lambda_4\beta_R}{\Lambda_2} \tag{2.83}$$

or

$$\Lambda_4 = -\frac{\Lambda_2}{\beta_R}\left[\delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2 - \delta_{oR}^{(o)}\right] \tag{2.84}$$

which is the desired condition for Λ_4^*.

It should be noted that the value of Λ_4 defined by Eq.2.84 corresponds to the real values of ε_{cr} and ε_{knee} only if

$$\frac{1}{\beta_R}\left[\delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2 - \delta_R^{(0)}\right] > 0 \tag{2.85}$$

It should also be noted that the fifth-order nonlinear term had an influence on (and only on) $\delta^{(2)}$ as is shown by Eq. 2.112 (which defines $\{H_3\}$), Eq. 2.119 (which defines δ_6), and Eq. 2.124 (which defines $\delta^{(2)}$). Thus, the fifth-order nonlinear terms will be called "destabilizing" if Eq. 2.85 is satisfied (that is, if they are such that the system will yield an unstable limit cycle for higher values of ε).

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*Combining Eqs. 2.75 and 2.84 yields $\delta_R^{(o)} = 2\delta_{o\,R}^{(o)} - \delta_R^{(2)}(\beta_R/\gamma_R)^2$ which simplifies considerably Eqs. 2.154 and 2.155.

2.6 Explicit Expressions for the Nonlinear Terms

In this subsection, explicit expressions for $\{C_3\}$, $\{C_5\}$, and $\{D_5\}$ are derived. Consider first, $\{C_3\}$. Combining Eqs. 2.14 and 2.20 yields

$$\{C_3\} = \begin{Bmatrix} C_{11}(Ae^{i\omega_F t_0} + A^* e^{-i\omega_F t_0})^3 + C_{12}(Ae^{i\omega_F t_0} + A^* e^{-i\omega_F t_0})^{2}(uAe^{i\omega_F t_0} + u^* A^* e^{-i\omega_F t_0})^{2} \\ C_{21}(Ae^{i\omega_F t_0} + A^* e^{-i\omega_F t_0})^2 (uAe^{i\omega_F t_0} + u^* A^* e^{-i\omega_F t_0}) + C_{22}(uAe^{i\omega_F t_0} + u^* A^* e^{-i\omega_F t_0})^3 \end{Bmatrix}$$

$$= 2\,\mathrm{Real}\left(A^3 \begin{Bmatrix} C_{11} + u^2 C_{12} \\ u C_{21} + u^3 C_{22} \end{Bmatrix} e^{3i\omega_F t_0} \right)$$

$$+ 2\,\mathrm{Real}\left(A^2 A^* \begin{Bmatrix} 3C_{11} + (u^2 + 2uu^*)\,C_{12} \\ (u^* + 2u)\,C_{21} + 3u^2 u^* C_{22} \end{Bmatrix} e^{i\omega_F t_0} \right)$$

$$= 2\,\mathrm{Real}\left(A^3 \{H_0\} e^{i3\omega_F t_0} \right) + 2\,\mathrm{Real}\left(A^2 A^* \{H_1\} e^{i\omega_F t_0} \right) \tag{2.86}$$

with

$$\{H_0\} = \begin{Bmatrix} C_{11} + u^2 C_{12} \\ u C_{21} + u^3 C_{22} \end{Bmatrix} \tag{2.87}$$

and

$$\{H_1\} = \begin{Bmatrix} 3C_{11} + (u^2 + 2uu^*)\,C_{12} \\ (u^* + 2u)\,C_{21} + 3u^2 u^* C_{22} \end{Bmatrix} \tag{2.88}$$

In the following, it is shown that ($\{C_5\}$ + $\{D_5\}$) is given by Eq. 2.107. In order to obtain this, it is convenient first to rewrite Eq. 2.36 in a slightly different form. First, combining Eqs. 2.24 and 2.26 yields

$$\{Z_3^{(1)}\} = \left[-\beta\,(2i\omega[I] + [G]) + \Lambda_2 [E] \right] \{u\} A$$

$$+ \left[-\gamma\,(2i\omega[I] + [G])\{u\} + \{H_1\} \right] A^2 A^* \tag{2.89}$$

Then combining Eqs. 2.37 and 2.89 yields

$$\{P_3^{(1)}\} = A\begin{Bmatrix} 0 \\ v \end{Bmatrix} \tag{2.90}$$

with

$$v = v^{(0)} + v^{(1)} A A^* \tag{2.91}$$

where

$$v^{(0)} = \frac{-1}{-\omega_F^2 + i g_2 \omega_F + \Omega_2^2} \lfloor 0,\ 1 \rfloor \left(-\beta\left(2i\omega[I] + [G]\right) + \Lambda_2[E]\right)\{u\}$$

$$v^{(1)} = \frac{-1}{-\omega_F^2 + i g_2 \omega_F + \Omega_2^2} \lfloor 0,\ 1 \rfloor \left[-\gamma\left(2i\omega[I] + [G]\right)\{u\} + \{H_1\}\right] \tag{2.92}$$

Similarly, combining Eqs. 2.23 and 2.38

$$\begin{aligned} \{P_3^{(3)}\} &= [L^{(3)}]\{H_0\} A^3 = \begin{Bmatrix} p_1 \\ p_3 \end{Bmatrix} A^3 \\ &= p_1 \begin{Bmatrix} 1 \\ p \end{Bmatrix} A^3 \end{aligned} \tag{2.93}$$

with $p = p_2/p_1$.

Thus, Eq. 2.36 can be rewritten as

$$\begin{Bmatrix} W_{13} \\ W_{23} \end{Bmatrix} = 2\,\mathrm{Real}\left[\left(B\begin{Bmatrix} 1 \\ u \end{Bmatrix} + A\begin{Bmatrix} 0 \\ v \end{Bmatrix}\right) e^{i\omega_F t_0} + p_1 A^3 \begin{Bmatrix} 1 \\ p \end{Bmatrix} e^{i3\omega_F t_0}\right] \tag{2.94}$$

Combining Eqs. 2.15, 2.20, and 2.94 yields

$$\{C_5\} = \begin{Bmatrix} \zeta_1 \\ \zeta_2 \end{Bmatrix} + \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} + \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix} \tag{2.95}$$

with

$$\zeta_1 = 2\,Real\,\Big[3A^2B\,(C_{11} + u^2C_{12})\,e^{i3\tau} + (A^2B^* + 2AA^*B)\,[3C_{11} + (u^2 + 2uu^*)C_{12}]\,e^{i\tau}\Big] \tag{2.96}$$

$$\zeta_2 = 2\,Real\,\Big[A^2B \cdot 3\,(u\,C_{21} + u^3C_{22})\,e^{3i\tau} + (2AA^*B + A^2B^*)\,[(2u + u^*)C_{21} + 3u^2u^*C_{22}]\,e^{i\tau}\Big] \tag{2.97}$$

$$\begin{aligned} \eta_1 &= 2\,Real\,\Big[\,C_{12}A^3\,2uv\,e^{3i\tau} + 2C_{12}A^2A^*(uv + u^*v + uv^*)\,e^{i\tau}\Big] \\ &= 2\,Real\,\Big[2A^3\,(u\,v\,C_{12})\,e^{i3\tau} + A^2A^*\,[2(u+u^*)v + 2uv^*]\,C_{12}\,e^{i\tau}\Big] \end{aligned} \tag{2.98}$$

$$\eta_2 = 2\,Real\,\Big[A^3\,(v\,C_{21} + u^2v\,C_{22})\,e^{3i\tau} + A^2A^*\,[(2v + v^*)C_{21} + 3(2uu^*v + u^2v^*)\,C_{22}]\,e^{i\tau}\Big] \tag{2.99}$$

$$\begin{aligned} \gamma_1 = 2\,Real\,\Big[&A^5p_1\,[3C_{11} + (u^2 + 2up)C_{12}]\,e^{i5\tau} + 2A^4A^*p_1\,[3C_{11} \\ &+ (uu^* + up + u^*p)\,C_{12}]\,e^{i3\tau} + A^2A^{*3}p_1^*\,[3C_{11} \\ &+ (u^2 + 2u\,p_1^*)\,C_{12}]\,e^{-i\tau}\Big] \end{aligned} \tag{2.100}$$

$$\begin{aligned} \gamma_2 = 2\,Real\,\Big[&A^5p_1\,[(2u + p)\,C_{21} + 3u^2pC_{22}\,]\,e^{i5\tau} \\ &+ 2A^4A^*p_1\,[(u + u^* + p)\,C_{21} + 3uu^*pC_{22}]\,e^{i3\tau} \\ &+ A^2A^{*3}p_1^*\,[(2u + p)\,C_{21} + 3u^2p^*C_4\,]\,e^{-i\tau}\Big] \end{aligned} \tag{2.101}$$

Summarizing

$$\{C_5\} = 2\,\mathrm{Real}\left[3A^2B \begin{Bmatrix} c_{11} + u^2 c_{12} \\ u c_{21} + u^3 c_{22} \end{Bmatrix} e^{i3\tau} + A^3 \begin{Bmatrix} u c_{12} \\ c_{21} + u^2 c_{22} \end{Bmatrix} v e^{i3\tau} \right.$$

$$+ (A^2B^* + 2AA^*B) \begin{Bmatrix} 3c_{11} + (u^2 + 2uu^*) c_{12} \\ (2u + u^*) c_{21} + 3u^2u^* c_{22} \end{Bmatrix} e^{i\tau}$$

$$+ A^2A^* \left(\begin{Bmatrix} (u+u^*) c_{12} \\ c_{21} + 3uu^* c_{22} \end{Bmatrix} 2v + \begin{Bmatrix} 2u c_{12} \\ c_{21} + 3u^2 c_{22} \end{Bmatrix} v^* \right) e^{i\tau}$$

$$+ A^5 \begin{Bmatrix} 3c_{11} + (u^2 + 2up) c_{12} \\ (2u + p) c_{21} + 3u^2 p c_{22} \end{Bmatrix} p_1 e^{i5\tau} + A^2A^{*3} \begin{Bmatrix} 3c_{11} + (u^2 + 2up^*) c_{12} \\ (2u + p^*) c_{21} + 3u^2 p^* c_{22} \end{Bmatrix} p_1^* e^{-i\tau}$$

$$\left. + 2A^4A^* \begin{Bmatrix} 3c_{11} + (uu^* + up + u^*p) c_{12} \\ (u + u^* + p) c_{21} + 3uu^*p c_{22} \end{Bmatrix} p_1 e^{i3\tau} \right] \qquad (2.102)$$

Next, consider $\{D_5\}$. Combining Eqs. 2.7 and 2.20 yields

$$\{D_5\} = \begin{Bmatrix} \bar{d}_{51} \\ \bar{d}_{52} \end{Bmatrix} \qquad (2.103)$$

with

$$\bar{d}_{51} = A^5 e^{i5\tau} (d_{11} + u^2 d_{12} + u^4 d_{13})$$

$$+ A^4A^* e^{i3\tau} \left[5d_{11} + (3u^2 + 2uu^*) d_{12} + (u^4 + 4u^3u^*) d_{13} \right]$$

$$+ A^3A^* e^{i\tau} \left[10d_{11} + (3u^2 + 6uu^* + u^{*2}) d_{12} \right.$$

$$\left. + (4u^3u^* + 6u^2u^{*2}) d_{13} \right] \qquad (2.104)$$

$$\bar{d}_{52} = A^5 e^{i5\tau}(u d_{21} + u^3 d_{22} + u^5 d_{23})$$
$$+ A^4 A^* e^{i3\tau}\left[(4u+u^*)d_{21} + (2u^3+3u^2u^*)d_{22} + 5u^4u^*d_{23}\right]$$
$$+ A^3 A^{*2} e^{i\tau}\left[(6u+4u^*)d_{21} + (u^3 + 6u^2u^* + 3uu^{*2})d_{22} + 10u^3u^{*2}d_{23}\right] \qquad (2.105)$$

Thus

$$\{D_5\} = A^5 e^{i5\tau}\begin{Bmatrix} d_{11} + u^2 d_{12} + u^4 d_{13} \\ d_{21}u + u^3 d_{22} + u^5 d_{23} \end{Bmatrix}$$
$$+ A^4 A^* e^{i3\tau}\begin{Bmatrix} 5d_{11} + (3u^2+2uu^*)d_{12} + (u^4+4u^3u^*)d_{13} \\ (4u+u^*)d_{21} + (2u^3+3u^2u^*)d_{22} + 5u^4u^*d_{23} \end{Bmatrix}$$
$$+ A^3 A^{*2} e^{i\tau}\begin{Bmatrix} 10d_{11} + (3u^2 + 6uu^* + u^{*2})d_{12} + (4u^3u^* + 6u^2u^{*2})d_{13} \\ (6u+4u^*)d_{21} + (u^3 + 6u^2u^* + 3uu^{*2})d_{22} + 10u^3u^{*2}d_{23} \end{Bmatrix}$$

(2.106)

Finally, combining Eqs. 2.102 and 2.106 yields

$$\{C_5\} + \{D_5\} = \{K_5^{(5)}\}e^{i5\tau} + \{K_5^{(3)}\}e^{i3\tau} + \{K_5^{(1)}\}e^{i\tau} \qquad (2.107)$$

with

$$\{K_5^{(5)}\} = A^5 \begin{Bmatrix} 3c_{11} + (u^2 + 2up)c_{12} \\ (2u+p)c_{21} + 3u^2 p c_{22} \end{Bmatrix} p_1$$
$$+ A^5 \begin{Bmatrix} d_{11} + u^2 d_{12} + u^4 d_{13} \\ u d_{21} + u^3 d_{22} + u^5 d_{23} \end{Bmatrix}$$

(2.108)

$$\left\{K_5^{(3)}\right\} = 3A^2B\begin{Bmatrix} c_{11}+u^2c_{12} \\ u\,c_{21}+u^3c_{22}\end{Bmatrix} + \left(v^{(0)}A^3 + v^{(1)}A^4A^*\right)\begin{Bmatrix} u\,c_{12} \\ c_{21}+u^2c_{22}\end{Bmatrix}$$

$$+ 2A^4A^*\begin{Bmatrix} 3c_{11}+[uu^*+(u+u^*)p]\,c_{12} \\ (u+u^*+p)\,c_{21}+3uu^*p\,c_{22}\end{Bmatrix} p_1$$

$$+ A^4A^*\begin{Bmatrix} 5d_{11}+(3u^2+2uu^*)d_{12}+(u^4+4u^3u^*)\,d_{13} \\ (4u+u^*)d_{21}+(2u^3+3u^2u^*)d_{22}+5u^4u^*d_{23}\end{Bmatrix} \qquad (2.109)$$

$$\left\{K_5^{(1)}\right\} = (A^2B^* + 2AA^*B)\begin{Bmatrix} 3c_{11}+(u^2+2uu^*)\,c_{12} \\ (2u+u^*)\,c_{21}+3u^2u^*c_{22}\end{Bmatrix}$$

$$+ \left(2v^{(0)}A^2A^* + 2v^{(1)}A^3A^{*2}\right)\begin{Bmatrix} (u+u^*)\,c_{12} \\ c_{21}+3uu^*c_{22}\end{Bmatrix} + \left(v^{(0)*}A^2A^* + v^{(1)*}A^3A^{*2}\right)\begin{Bmatrix} 2u\,c_{12} \\ c_{21}+3u^2c_{22}\end{Bmatrix}$$

$$+ A^3A^{*2}\begin{Bmatrix} 3c_{11}+(u^{*2}+2u^*p)\,c_{12} \\ (2u^*+p)\,c_{21}+3u^{*2}p\,c_{22}\end{Bmatrix} p_1$$

$$+ A^3A^{*2}\begin{Bmatrix} 10d_{11}+(3u^2+6uu^*+u^{*2})d_{12}+(4u^3u^*+6u^2u^{*2})\,d_{13} \\ (6u+4u^*)d_{21}+(u^3+6u^2u^*+3uu^{*2})d_{22}+10u^3u^{*2}d_{23}\end{Bmatrix}$$

$$= (A^2B^* + 2AA^*B)\{H_1\} + A^2A^*\{H_2\} + A^3A^{*2}\{H_3\} \qquad (2.110)$$

with $\{H_1\}$ given by Eq. 2.88 and

$$\{H_2\} = \begin{Bmatrix} (u+u^*)\,c_{12} \\ c_{21}+3uu^*c_{22}\end{Bmatrix} 2v^{(0)} + \begin{Bmatrix} 2u\,c_{12} \\ c_{21}+3u^2c_{22}\end{Bmatrix} v^{(0)*} \qquad (2.111)$$

$$\{H_3\} = \begin{Bmatrix} 3c_{11}+(u^{*2}+2u^*p)c_{12} \\ (2u^*+p)c_{21}+3u^{*2}p\,c_{22}\end{Bmatrix} p_1 + \begin{Bmatrix} (u+u^*)\,c_{12} \\ c_{21}+3uu^*c_{22}\end{Bmatrix} 2v^{(1)} + \begin{Bmatrix} 2u\,c_{12} \\ c_{21}+3u^2c_{22}\end{Bmatrix} v^{(1)*}$$

$$+ \begin{Bmatrix} 10d_{11}+(3u^2+6uu^*+u^{*2})d_{12}+(4u^3u^*+6u^2u^{*2})\,d_{13} \\ (6u+4u^*)d_{21}+(u^3+6u^2u^*+3uu^{*2})\,d_{22}+10u^3u^{*2}d_{23}\end{Bmatrix} \qquad (2.112)$$

2.7 Mathematical Elaborations

In order to simplify the discussion of the condition for avoiding secular terms (Subsection 2.5), most of the mathematical elaborations involved in the solution for Eq. 2.49 are described in the three subsections which follow.

2.7.1 Explicit Expression for $\{Z_5^{(1)}\}$

As shown in Subsection 2.4, the condition for avoiding secular terms in the solution for $\{W_5\}$ is given by Eq. 2.49:

$$\lfloor U^b \rfloor \{ Z_5^{(1)} \} = 0 \tag{2.113}$$

In this subsection, the explicit expression for $\{Z_5^{(1)}\}$ is derived and then combined with Eq. 2.113 in order to obtain the differential equation for B.

Combining Eqs. 2.37, 2.48, and 2.110 yields

$$\begin{aligned} \{Z_5^{(1)}\} = {} & [S]\{u\}\frac{\partial B}{\partial t_2} + \Lambda_2[E]\{u\}B \\ & + \left([S]\frac{\partial}{\partial t_2} + \Lambda_2[E]\right)[N]\{Z_3^{(1)}\} \\ & + [S]\{u\}\frac{\partial A}{\partial t_4} + \{u\}\frac{\partial^2 A}{\partial t_2^2} + \Lambda_4[E]\{u\}A \\ & + (A^2B^* + 2AA^*B)\{H_1\} + A^2A^*\{H_2\} \\ & + A^3A^{*2}\{H_3\} \end{aligned} \tag{2.114}$$

with

$$[S] = 2i\omega[I] + [G] \tag{2.115}$$

By using Eq. 2.24, Eq. 2.114 reduces to

$$\begin{aligned}\{Z_5^{(1)}\} &= [S]\{u\}\frac{\partial B}{\partial t_2} + \Lambda_2[E]\{u\}B + \{H_1\}(A^2B^* + 2AA^*B) \\ &+ \left([S][N][S]\{u\} + \{u\}\right)\frac{\partial^2 A}{\partial t_2^2} \\ &+ \Lambda_2\left([S][N][E] + [E][N][S]\right)\{u\}\frac{\partial A}{\partial t_2} \\ &+ \left(\Lambda_2^2[E][N][E] + \Lambda_4[E]\right)\{u\}A \\ &+ [S][N]\{H_1\}\frac{\partial}{\partial t_2}(A^2A^*) + \{H_3\}A^3A^{*2} + [S]\{u\}\frac{\partial A}{\partial t_4} \\ &+ \left(\Lambda_2[E][N]\{H_1\} + \{H_2\}\right)A^2A^*\end{aligned} \tag{2.116}$$

2.7.2 Equation for B

Combining Eqs. 2.54 and 2.116 yields

$$\alpha\left[\frac{\partial B}{\partial t_2} + \beta B + \gamma(A^2B^* + 2AA^*B)\right] + \delta = 0 \tag{2.117}$$

with α, β and γ given by Eqs. 2.29, 2.27, and 2.28, and with δ given by

$$\begin{aligned}\delta &= \delta_1\frac{\partial^2 A}{\partial t_2^2} + \delta_2\frac{\partial A}{\partial t_2} + \delta_3 A + \delta_4\frac{\partial}{\partial t_2}(A^2A^*) \\ &+ \delta_5 A^2A^* + \delta_6 A^3A^{*2} + \alpha\frac{\partial A}{\partial t_4}\end{aligned} \tag{2.118}$$

with

$$\delta_1 = \lfloor U^L \rfloor \left([S][N][S] + [I] \right) \{U\}$$
$$\delta_2 = \Lambda_2 \lfloor U^L \rfloor \left([S][N][E] + [E][N][S] \right) \{U\}$$
$$\delta_3 = \lfloor U^L \rfloor \left(\Lambda_2^2 [E][N][E] + \Lambda_4 [E] \right) \{U\}$$
$$= \Lambda_2^2 \lfloor U^L \rfloor [E][N][E]\{U\} + \Lambda_4 \alpha \beta / \Lambda_2$$
$$\delta_4 = \lfloor U^L \rfloor [S][N]\{H\}$$
$$\delta_5 = \lfloor U^L \rfloor \left(\Lambda_2 [E][N]\{H_1\} + \{H_2\} \right)$$
$$\delta_6 = \lfloor U^L \rfloor \{H_3\} \tag{2.119}$$

Next, a simpler expression for δ is derived, by using the expression for $\partial A/\partial t_2$ given by Eq. 2.26

$$\frac{\partial A}{\partial t_2} + \beta A + \gamma A^2 A^* = 0 \tag{2.120}$$

Combining Eqs. 2.118 and 2.120 yields

$$\delta = (\delta_2 - \delta_1 \beta) \frac{\partial A}{\partial t_2} + \delta_3 A + (\delta_4 - \delta_1 \gamma) \frac{\partial}{\partial t_2}(A^2 A^*)$$
$$+ \delta_5 A^2 A^* + \delta_6 A^3 A^{*2} + \alpha \frac{\partial A}{\partial t_4} \tag{2.121}$$

Next note that*

$$\frac{\partial}{\partial t_2}(A^2A^*) + (2\beta+\beta^*)A^2A^* + (2\gamma+\gamma^*)A^3A^* = 0 \tag{2.122}$$

By using Eqs. 2.120 and 2.122, Eq. 2.121 reduces to

$$\delta = \left[\delta^{(0)} + \delta^{(1)}AA^* + \delta^{(2)}(AA^*)^2 + \frac{\partial}{\partial t_4}\ln A\right]\alpha A \tag{2.123}$$

with

$$\delta^{(0)} = \frac{1}{\alpha}\left[\delta_3 - (\delta_2 - \delta_1\beta)\beta\right]$$

$$\delta^{(1)} = \frac{1}{\alpha}\left[\delta_5 - (\delta_2 - \delta_1\beta)\gamma - (\delta_4 - \delta_1\gamma)(2\beta+\beta^*)\right]$$

$$\delta^{(2)} = \frac{1}{\alpha}\left[\delta_6 - (\delta_4 - \delta_1\gamma)(2\gamma+\gamma^*)\right]$$

(2.124)

---

*In fact, using Eq. 2.120

$$\frac{\partial}{\partial t_2}(A^2A^*) = 2A\frac{\partial A}{\partial t_2}A^* + A^2\frac{\partial A^*}{\partial t_2}$$

$$= -2AA^*(\beta A + \gamma A^2A^*) - A^2(\beta^*A^* + \gamma^*A^{*2}A)$$

$$= -(2\beta+\beta^*)A^2A^* - (2\gamma+\gamma^*)A^3A^{*2}$$

2.7.3 The Functions b_R and b_I

Note, first, that Eq. 2.61 is equivalent to*

$$(\beta_R+\gamma_R|A|^2)\frac{\partial}{\partial t_2}\left(\frac{b_R}{\beta_R+\gamma_R|A|^2}\right)+\left(\frac{\delta}{\alpha A}\right)_R=0 \tag{2.125}$$

By the use of Eq. 2.56, Eq. 2.125 reduces to**

$$\frac{\partial}{\partial t_2}\left(\frac{b_R}{\beta_R+\gamma_R|A|^2}\right)=-\frac{1}{\beta_R+\gamma_R|A|^2}\Big[\delta_R^{(0)}+\delta_R^{(1)}|A|^2+ \\ +\delta_R^{(2)}|A|^4+\frac{\partial}{\partial t_4}\ln|A|\Big] \tag{2.126}$$

Next, note that, according to Eq. 2.31

$$\frac{1}{|A|^2}=-\frac{\gamma_R}{\beta_R}+K\,e^{2\beta_R t_2} \tag{2.127}$$

with K a function of t_4. Thus

---

*In fact, using Eq. 2.41 yields

$$(\beta_R+\gamma_R|A|^2)\frac{\partial}{\partial t_2}\left(\frac{b_R}{\beta_R+\gamma_R|A|^2}\right)=\frac{\partial b_R}{\partial t_2}-\frac{2\,b_R\gamma_R}{\beta_R+\gamma_R|A|^2}|A|\frac{\partial|A|}{\partial t_2}=$$

$$=\frac{\partial b_R}{\partial t_2}+2\gamma_R|A|^2 b_R$$

**Note that $\ln A=\ln|A|+j(\phi+2\pi k)$

$$\frac{\partial}{\partial t_4} \ln|A| = -\frac{1}{2}\frac{\partial}{\partial t_4} \ln\frac{1}{|A|^2} = -\frac{1}{2}|A|^2\frac{\partial}{\partial t_4}\frac{1}{|A|^2}$$

$$= -\frac{1}{2}|A|^2\frac{\partial K}{\partial t_4}\, e^{2\beta_R t_2} \tag{2.128}$$

and

$$\beta_R + \gamma_R|A|^2 = |A|^2\beta_R\left(\frac{1}{|A|^2} + \frac{\gamma_R}{\beta_R}\right) = |A|^2\beta_R K e^{2\beta_R t_2} \tag{2.129}$$

Combining Eqs. 2.126, 2.128, and 2.129 yields

$$\frac{\partial}{\partial t_2}\left(\frac{b_R}{\beta_R + \gamma_R|A|^2}\right) = \frac{-e^{-2\beta_R t_2}}{\beta_R K}\left[\delta_R^{(0)}|A|^{-2} + \delta_R^{(1)}\right.$$

$$\left. + \delta_R^{(2)}|A|^2\right] + \frac{1}{2\beta_R K}\frac{\partial K}{\partial t_4} \tag{2.130}$$

$$\frac{\partial}{\partial t_2}\left(\frac{b_R}{\beta_R + \gamma_R|A|^2}\right) = -\frac{\delta_R^{(0)}}{\beta_R K}\, e^{-2\beta_R t_2}\left(-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2}\right)$$

$$-\frac{\delta_R^{(1)}}{\beta_R K}\, e^{-2\beta_R t_2} + \frac{1}{2\beta_R K}\frac{\partial K}{\partial t_4} \tag{2.131}$$

$$-\frac{\delta_R^{(2)}}{\beta_R K}\left(\frac{e^{-2\beta_R t_2}}{-\frac{\gamma_R}{\beta_R} + K e^{2\beta_R t_2}}\right)$$

Integrating Eq. 2.131 yields

$$\frac{b_R}{\beta_R+\gamma_R|A|^2} = -\frac{\delta_R^{(0)}}{\beta_R K}\left[\frac{\gamma_R e^{-2\beta_R t_2}}{2\beta_R^2} + K t_2\right] + \frac{\delta_R^{(1)}}{\beta_R K}\left(\frac{e^{-2\beta_R t_2}}{2\beta_R}\right)$$

$$-\frac{\delta_R^{(2)}}{\beta_R K}\left[\frac{1}{2\gamma_R} e^{-2\beta_R t_2} - K\left(\frac{\beta_R}{\gamma_R}\right)^2 t_2 + \frac{K\beta_R}{2\gamma_R^2}\ln\left(-\frac{\gamma_R}{\beta_R}+Ke^{2\beta_R t}\right)\right]$$

$$+\frac{1}{2\beta_R K}\frac{\partial K}{\partial t_4} t_2 + \frac{b_R^{(0)}}{\beta_R} \qquad (2.132)$$

where $b_R^{(o)}$ is a constant of integration. Equation 2.132 can be rewritten, more conveniently, as

$$\frac{b_R}{\beta_R+\gamma_R|A|^2} = -\frac{\gamma_R}{2\beta_R^3 K}\left[\delta_R^{(0)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] e^{-2\beta_R t_2}$$

$$-\delta_R^{(2)}\frac{1}{2\gamma_R^2}\ln\left(-\frac{\gamma_R}{\beta_R} + Ke^{2\beta_R t_2}\right)$$

$$+\frac{1}{\beta_R}\left[\frac{1}{2K}\frac{\partial K}{\partial t_4} - \delta_R^{(0)} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] t_2 + \frac{b_R^{(0)}}{\beta_R} \qquad (2.133)$$

In the following, only, the case $\gamma_R > 0$ is discussed. Then the limit cycle exists only for $\Lambda_2 = 1$, or (see Ref. 2)

$$\beta_R < 0 \qquad (2.134)$$

Then the logarithm term goes to $n(-\gamma_R/\beta_R)$ as t_2 goes to infinity.* Thus, in order to avoid secular terms, the following condition must be satisfied

$$\frac{1}{2K}\frac{\partial K}{\partial t_4} - \delta_R^{(0)} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2 = 0 \qquad (2.135)$$

or

$$K = K_0\, e^{2\left[\delta_R^{(0)} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] t_4} \qquad (2.136)$$

---

*The discussion for $\gamma_R < 0$ is modified as follows: the case $\beta_R < 0$ ($\Lambda_2 = -1$) is considered and the limit is obtained as $t_2 \to -\infty$ (unstable limit cycle). The results are the same as those for $\gamma_R > 0$.

Finally, combining Eqs. 2.41, 2.129, 2.133, and 2.135 yields

$$b_R = |A|^2 \beta_R K e^{2\beta_R t_2} \frac{(-\gamma_R)}{2\beta_R^3 K} \left[\delta_R^{(0)} - \delta_R^{(1)} \frac{\beta_R}{\gamma_R} + \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R}\right)^2 \right] e^{-2\beta_R t_2}$$

$$- (\beta_R + \gamma_R |A|^2) \left(\frac{\delta_R^{(2)}}{2\gamma_R^2} \ln |A|^{-2} - \frac{b_R^{(0)}}{\beta_R} \right) \qquad (2.137)$$

or

$$b_R = \chi_2 |A|^2 - (\chi_0 + \chi_1 \ln |A|)(\beta_R + \gamma_R |A|^2) \qquad (2.138)$$

with

$$\chi_0 = - \frac{b_R^{(0)}}{\beta_R}$$

$$\chi_1 = - \frac{\delta_R^{(2)}}{\gamma_R^2}$$

$$\chi_2 = - \frac{\gamma_R}{2\beta_R^2} \left[\delta_R^{(0)} - \delta_R^{(1)} \frac{\beta_R}{\gamma_R} + \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] \qquad (2.139)$$

Consider next the function b_I. Combining Eqs. 2.62 and 2.56 yields

$$\frac{\partial b_I}{\partial t_2} = -2\gamma_I |A|^2 b_R - \Big(\delta_I^{(0)} + \delta_I^{(1)} |A|^2 + \delta_I^{(2)} |A|^4 + \frac{\partial \varphi}{\partial t_4} \Big) \qquad (2.140)$$

In order to find a simple expression for $\partial\phi/\partial t_4$, note first that combining Eqs. 2.128, 2.129, 2.31 and 2.135 yields

$$\frac{\partial}{\partial t_4} \ln |A| = -\frac{1}{2} |A|^2 e^{2\beta_R t} \frac{\partial K}{\partial t_4} = -\frac{1}{2} \frac{\partial K}{\partial t_4} \frac{\beta_R + \gamma_R |A|^2}{\beta_R K}$$

$$= \frac{1}{2\beta_R K} \frac{\partial K}{\partial t_4} \frac{1}{|A|} \frac{\partial |A|}{\partial t_2} = \frac{1}{\beta_R} \left[\delta_R^{(0)} - \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] \frac{1}{|A|} \frac{\partial |A|}{\partial t_2} \qquad (2.141)$$

Thus, according to Eqs. 2.33 and 2.141

$$\frac{\partial \varphi}{\partial t_4} = \frac{\gamma_I}{\gamma_R} \frac{\partial}{\partial t_4} \ln |A| + \frac{\partial \varphi_0}{\partial t_4}$$

$$= \frac{\gamma_I}{\gamma_R \beta_R} \left[\delta_R^{(0)} - \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] \frac{1}{|A|} \frac{\partial |A|}{\partial t_2} + \frac{\partial \varphi_0}{\partial t_4} \tag{2.142}$$

Thus combining Eqs. 2.138, 2.140, and 2.142 yields

$$\frac{\partial b_I}{\partial t_2} = -2\gamma_I (\chi_0 + \chi_1 \ln |A|) |A| \frac{\partial |A|}{\partial t_2} - 2\gamma_I \chi_2 |A|^4$$

$$- (\delta_I^{(0)} + \delta_I^{(1)} |A|^2 + \delta_I^{(2)} |A^4|)$$

$$- \frac{\gamma_I}{\gamma_R \beta_R} \left[\delta_R^{(0)} - \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] \frac{1}{|A|} \frac{\partial |A|}{\partial t_2} - \frac{\partial \varphi_0}{\partial t_4} \tag{2.143}$$

or

$$\frac{\partial b_I}{\partial t_2} = \eta_0 + \eta_1 |A|^2 + \eta_2 |A|^4 + \eta_3 \frac{\partial |A|^2}{\partial t_2}$$

$$+ \eta_4 \ln |A|^2 \frac{\partial |A|^2}{\partial t_2} + \eta_5 \frac{\partial}{\partial t_2} \ln |A| \tag{2.144}$$

with

$$\eta_0 = -\delta_I^{(0)} - \frac{\partial \varphi_0}{\partial t_4}$$

$$\eta_1 = -\delta_I^{(1)}$$

$$\eta_2 = -\delta_I^{(2)} - 2\gamma_I \chi_2$$

$$\eta_3 = -\gamma_I \chi_0$$

$$\eta_4 = -\frac{1}{2} \gamma_I \chi_1$$

$$\eta_5 = \frac{-\gamma_I}{\gamma_R \beta_R} \left[\delta_R^{(0)} - \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] \tag{2.145}$$

Integrating Eq. 2.144 yields

$$b_I = \eta_0 t_2 - \eta_1 \left(\frac{\beta_R}{\gamma_R} t_2 + \frac{1}{\gamma_R} \ln |A| \right)$$
$$+ \eta_2 \left[\frac{\beta_R^2}{\gamma_R^2} t_2 + \frac{\beta_R}{\gamma_R^2} \ln |A| - \frac{1}{2\gamma_R} |A|^2 \right]$$
$$+ \eta_3 |A|^2 + \eta_4 \left(|A|^2 \ln |A|^2 - |A|^2 \right) + \eta_5 \ln A + b_I^{(0)} \tag{2.146}$$

or

$$b_I = \left(\eta_0 - \eta_1 \frac{\beta_R}{\gamma_R} + \eta_2 \frac{\beta_R^2}{\gamma_R^2} \right) t_2 + \left(- \frac{\eta_1}{\gamma_R} + \eta_2 \frac{\beta_R}{\gamma_R^2} + \eta_5 \right) \ln |A|$$
$$+ \left(- \frac{\eta_2}{2\gamma_R} + \eta_3 - \eta_4 \right) |A|^2 + 2\eta_4 |A|^2 \ln |A| + b_I^{(0)} \tag{2.147}$$

In order to avoid secular terms in b_I, the following condition must be satisfied

$$\eta_0 - \eta_1 \frac{\beta_R}{\gamma_R} + \eta_2 \frac{\beta_R^2}{\gamma_R^2} = 0 \tag{2.148}$$

or, according to Eqs. 2.139 and 2.145

$$- \frac{\partial \varphi_0}{\partial t_4} - \delta_I^{(0)} + \delta_I^{(1)} \frac{\beta_R}{\gamma_R} - \delta_I^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2$$
$$+ \frac{\gamma_I}{\gamma_R} \left[\delta_R^{(0)} - \delta_R^{(1)} \frac{\beta_R}{\gamma_R} + \delta_R^{(2)} \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] = 0 \tag{2.149}$$

or

$$\varphi_0 = \bar{\Phi}_1 t_4 + \bar{\Phi}_0 \tag{2.150}$$

where ϕ_o is an arbitrary constant and

$$\begin{aligned}\Phi_1 = &-\left[\delta_I^{(0)} - \delta_I^{(1)}\frac{\beta_R}{\gamma_R} + \delta_I^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] \\ &+ \frac{\gamma_I}{\gamma_R}\left[\delta_R^{(0)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\end{aligned} \tag{2.151}$$

Summarizing, b_R and b_I are given by (see Eqs. 2.138, 2.147, and 2.148)

$$\begin{aligned}b_R = \; &b_R^{(0)} + b_R^{(1)}|A|^2 + b_R^{(2)}\ln|A| \\ &+ b_R^{(3)}|A|^2\ln|A|\end{aligned} \tag{2.152}$$

$$\begin{aligned}b_I = \; &b_I^{(0)} + b_I^{(1)}|A|^2 + b_I^{(2)}\ln|A| \\ &+ b_I^{(3)}|A|^2\ln|A|\end{aligned} \tag{2.153}$$

where $b_R^{(o)}$ and $b_I^{(o)}$ are two arbitrary constants and (see Eq. 2.139)

$$\begin{aligned}b_R^{(1)} &= \chi_2 - \chi_0\gamma_R \\ &= -\frac{\gamma_R}{2\beta_R^2}\left[\delta_R^{(0)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] + b_R^{(0)}\frac{\gamma_R}{\beta_R} \\ b_R^{(2)} &= -\chi_1\beta_R = \frac{\beta_R}{\gamma_R^2}\,\delta_R^{(2)} \\ b_R^{(3)} &= -\gamma_R\chi_1 = \frac{1}{\gamma_R}\,\delta_R^{(2)}\end{aligned} \tag{2.154}$$

whereas (see Eqs. 2.147 and 2.145)

$$b_I^{(1)} = -\frac{\eta_2}{2\gamma_R} + \eta_3 - \eta_4 = -\frac{1}{2\gamma_R}\left(-\delta_I^{(2)} - 2\gamma_I \chi_2\right) - \gamma_I \chi_0 + \frac{1}{2}\gamma_I \chi_1$$

$$= -\frac{1}{2\gamma_R}\left\{-\delta_I^{(2)} + \frac{\gamma_I \gamma_R}{\beta_R^2}\left[\delta_R^{(0)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\right\} + \frac{\gamma_I}{\beta_R} b_R^{(0)} - \frac{\gamma_I}{2\gamma_R^2}\delta_R^{(2)}$$

$$= -\frac{\gamma_I}{2\beta_R^2}\left[\delta_R^{(0)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right] + \frac{\gamma_I}{\beta_R} b_R^{(0)} + \frac{1}{2\gamma_R}\left(\delta_I^{(2)} - \frac{\gamma_I}{\gamma_R}\delta_R^{(2)}\right)$$

$$= \frac{\gamma_I}{\gamma_R} b_R^{(1)} + \frac{1}{2\gamma_R}\left(\delta_I^{(2)} - \frac{\gamma_I}{\gamma_R}\delta_R^{(2)}\right)$$

$$b_I^{(2)} = -\frac{\eta_1}{\gamma_R} + \eta_2\frac{\beta_R}{\gamma_R^2} + \eta_5$$

$$= \frac{\delta_I^{(1)}}{\gamma_R} + \frac{\beta_R}{\gamma_R^2}\left(-\delta_I^{(2)} - 2\gamma_I \chi_2\right) - \frac{\gamma_I}{\beta_R \gamma_R}\left[\delta_R^{(0)} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right]$$

$$= \frac{\delta_I^{(1)}}{\gamma_R} - \frac{\beta_R}{\gamma_R^2}\delta_I^{(2)} + \frac{\gamma_I}{\gamma_R \beta_R}\left[\delta_R^{(0)} - \delta_R^{(1)}\frac{\beta_R}{\gamma_R} + \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right]$$

$$- \frac{\gamma_I}{\beta_R \gamma_R}\left[\delta_R^{(0)} - \delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2\right]$$

$$= \frac{1}{\gamma_R}\left[\delta_I^{(1)} - \frac{\beta_R}{\gamma_R}\delta_I^{(2)}\right] + \frac{\gamma_I}{\beta_R \gamma_R}\left[2\delta_R^{(2)}\left(\frac{\beta_R}{\gamma_R}\right)^2 - \delta_R^{(1)}\frac{\beta_R}{\gamma_R}\right]$$

$$b_I^{(3)} = 2\eta_4 = -\gamma_I \chi_1 = \frac{\gamma_I}{\gamma_R} b_R^{(3)} \tag{2.155}$$

SECTION 3

FLUTTER-BUCKLING INTERACTION

3.1 Introduction

As mentioned in the Introduction, the analysis given in Section 2 is valid only if $\omega_F > 0$. Thus, the results obtained in Section 2 are not valid in the neighborhood of the intersection point (intersection of the flutter boundary with the buckling boundary, defined in Subsection A.7*) for which $\omega = 0$ is a double root of the characteristic equation. For this section, a new analysis is developed in order to study the behavior of the system in the neighborhood of the intersection point where

$$N = N_* \quad \text{and} \quad \Lambda = \Lambda_* \tag{3.1}$$

with N_* and Λ_* given by Eq. A.65; the quantity N is the applied membrane force and Λ is the dynamic pressure parameter (see Eq. 1.10).

The behavior of the system in the neighborhood of this intersection point can be studied by setting

$$\Lambda = \Lambda_* + \varepsilon^2 \sin\theta$$
$$N = N_* + \varepsilon^2 \cos\theta \tag{3.2}$$

where θ is an arbitrary parameter, which gives the direction in the plane (Λ, N) in which the limit $\varepsilon \to 0$ is considered (see Fig. 2). Note that Eq. A.3 can be written as

$$\Omega_1^2 = \Omega_{10}^2 \left[1 - \frac{1}{N_1} \left(N_* + \varepsilon^2 \cos\theta \right) \right] = \Omega_{1*}^2 + \Delta_1 \varepsilon^2 \cos\theta$$
$$\Omega_2^2 = \Omega_{20}^2 \left[1 - \frac{1}{N_2} \left(N_* + \varepsilon^2 \cos\theta \right) \right] = \Omega_{2*}^2 + \Delta_2 \varepsilon^2 \cos\theta \tag{3.3}$$

with

$$\Omega_{i*}^2 = \Omega_{i0}^2 \left(1 - \frac{N_*}{N_i} \right)$$
$$\Delta_i = - \frac{\Omega_{i0}^2}{N_i} \tag{3.4}$$

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*The intersection point is obtained by studying the linear problem. For the sake of simplicity, the linear case is fully discussed in Appendix A.

A summary of the results obtained in this section is given in the following. In matrix form, the equation to be solved is given by Eq. 3.5. By introducing the multiple scales $t_n = \varepsilon^n t$ (Eq. 3.7), and by assuming an asymptotic expansion for $\{W\}$ (Eq. 3.8), one obtains a system given by Eq. 3.9. Separating terms of the same order yields the set of systems given by Eqs. 3.14 to 3.17. The solution of the first-order system contains secular terms (Eq. A.68) which are dropped*. If the damped terms (see Subsection A.7) are also dropped, the solution for $\{W_1\}$ reduces to Eq. 3.18. Note that this solution does not depend upon t_o (but the damped part of the solution does depend upon t_o).

Next, the second order system is considered. It is easily shown that the condition for no secular terms is automatically satisfied.** Thus, the dependence of $\{W_1\}$ on t_1 cannot be determined at this step (the analysis of the third-order system is necessary). The vector $\{W_2\}$ is given by Eq. 3.25.

The third-order system is then considered in Subsection 3.4. In order to avoid secular terms, Eq. 3.41 must be satisfied and then the vector $\{W_2\}$ is given by Eq. 3.45. The condition for no secular terms (Eq. 3.41) is a differential equation for A (coefficient introduced in the solution for $\{W_1\}$, see Eq. 3.18) as a function of t_1. The solution of this equation is discussed in Subsection 3.5 for $\gamma > 0$ (hard spring nonlinear terms) only. This is given by either Eq. 3.53 (vibration about the flat position: <u>unbuckled case</u>) or by Eq. 3.58 (vibration about the buckled position: <u>buckled case</u>) depending upon the values of the inplane force (which affect the value of β and the energy $\mathcal{E}$ (Eq. 3.48). For both cases the solution is represented by a periodic function elliptic Jacobian functions (whose properties are discussed in Subsection 3.10), with unknown "amplitude".

Thus, in order to study the transient and the limit cycle solutions,

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*This is "compensated for" by the introduction of the time scales t_1, t_3, ... , and the vectors $\{W_2\}$, $\{W_4\}$ which do not exist in the basic problem treated in Section 2.

**This is due to the fact that $[U^L]$ $[G]$ $\{U\} = 0$ (Eq. 3.28); because of the importance of this relation in this analysis, the final part of Subsection 3.3 is devoted to proving that Eq. 3.28 is valid for the N-mode case also.

it is necessary to consider the fourth-order system, given by Eq. 3.62. If the condition of no secular terms, $\lfloor U^L \rfloor \{Z_4\} = 0$ is satisfied, the solution $\{W_4\}$ is given by Eq. 3.68. On the other hand, the condition for no secular terms yields a differential equation for B (coefficient introduced in the solution for W_2, see Eq. 3.25) as a function of t_1, given by Eq. 3.66. The solution of this equation is discussed in Subsection 3.7, where it is shown that this equation yields secular terms of the type t_1^2 and t_1. It is also shown that avoiding terms of type t_1^2 yields the variation of A with t_2, but then it is impossible to avoid secular terms of the type t_1, unless the steady-state solution ($\partial A_o/\partial t_2 = 0$) is considered. In this case, terms of type t_1^2 are eliminated by assuming that the amplitude A_o is such that Eq. 3.85 is satisfied (then the secular terms of the type t_1 can be easily eliminated). Thus, summarizing, by satisfying Eq. 3.85 one obtains the amplitude A_o of the limit cycle but the transient response cannot be studied without introducing secular terms in the solution for B. A very similar situation is encountered in Section 4 where it is easier to interpret the results in terms of the results obtained in Section 2. An attempt at interpretation is given in Section 5.

3.2 General Formulation

Combining Eqs. A.1 and 3.2 yields (in matrix notation)

$$\{\ddot{W}\} + [G]\{\dot{W}\} + \left([\Omega_*^2] + \varepsilon^2 \cos\theta [\Delta] \right) W$$
$$+ (\Lambda_* + \varepsilon^2 \sin\theta)[E]\{W\} + \{C\} + \cdots = 0 \qquad (3.5)$$

where the symbols given by Eqs. 3.2 and 3.3 have been used, and furthermore

$$[\Omega_*^2] = \lceil \Omega_{i*}^2 \rfloor$$
$$[\Delta] = \lceil \Delta_i \rfloor \qquad (3.6)$$

It should be noted that the solution of the linear system at the intersection point contains a secular term (see Eq. A.68). In order to obtain a solution which does not contain this secular term, it is necessary to introduce intermediate time scales ($t_1 = t$, $t_3 = \varepsilon^3 t$ . . .) which were not used in Section 2. Thus, introduce the multiple time scales

$$t_n = \varepsilon^n t \qquad (n = 0, 1, 2 \ldots) \tag{3.7}$$

Consequently the appropriate expansion for $\{W\}$ is given by

$$\{W\} = \varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \varepsilon^3\{W_3\} + \varepsilon^4\{W_4\} + \cdots \tag{3.8}$$

Note that the vectors $\{W_2\}$ and $\{W_4\}$ are identically equal to zero in the analysis given in Section 2. Finally, combining Eqs. 3.5 through 3.8 yields

$$\begin{aligned}&\left(\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \cdots\right)^2 (\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \cdots) + [G] \\ &\times \left(\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \cdots\right)(\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \cdots) + ([F_0] + \varepsilon^2[F_2]) \\ &\times (\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \cdots) + \varepsilon^3\{C_3\} + \varepsilon^4\{C_4\} + \cdots = 0\end{aligned} \tag{3.9}$$

with

$$[F_0] = [\Omega_*^2] + \Lambda_*[E] \tag{3.10}$$

$$[F_2] = \cos\theta\,[\Delta] + \sin\theta\,[E] \tag{3.11}$$

and

$$\{C_3\} = \begin{Bmatrix} C_{11} W_{11}^3 + C_{12} W_{11} W_{21}^2 \\ C_{21} W_{11}^2 W_{21} + C_{22} W_{21}^3 \end{Bmatrix} \tag{3.12}$$

$$\{C_4\} = \begin{Bmatrix} 3C_{11} W_{11}^2 W_{12} + C_{12}(W_{12} W_{21}^2 + 2W_{11} W_{21} W_{22}) \\ C_{21}(2W_{11} W_{12} W_{21} + W_{11}^2 W_{22}) + 3C_{22} W_{21}^2 W_{22} \end{Bmatrix} \tag{3.13}$$

Separating terms of the same order of magnitude yields the following set of systems:

Order ε:

$$\frac{\partial^2\{W_1\}}{\partial t_0^2} + [G]\frac{\partial\{W_1\}}{\partial t_0} + [F_0]\{W_1\} = 0 \tag{3.14}$$

Order ε^2:

$$\frac{\partial^2\{W_2\}}{\partial t_0^2} + [G]\frac{\partial\{W_2\}}{\partial t_0} + [F_0]\{W_2\} + \\ + 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_1\} + [G]\frac{\partial}{\partial t_1}\{W_1\} = 0 \tag{3.15}$$

Order ε^3:

$$\frac{\partial^2\{W_3\}}{\partial t_0^2} + [G]\frac{\partial}{\partial t_0}\{W_3\} + [F_0]\{W_3\} + \\ + 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_2\} + [G]\left(\frac{\partial}{\partial t_1}\{W_2\} + \frac{\partial}{\partial t_2}\{W_1\}\right) \\ + \left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right)\{W_1\} + [F_2]\{W_1\} + \{C_3\} = 0 \tag{3.16}$$

Order ε^4:

$$\frac{\partial^2}{\partial t_0^2}\{W_4\} + [G]\frac{\partial}{\partial t_0}\{W_4\} + [F_0]\{W_4\} + \\ + 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_3\} + \left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right)\{W_2\} + \left(2\frac{\partial^2}{\partial t_0 \partial t_3} + 2\frac{\partial^2}{\partial t_1 \partial t_2}\right)\{W_1\} \\ + [G]\left(\frac{\partial}{\partial t_1}\{W_3\} + \frac{\partial}{\partial t_2}\{W_2\} + \frac{\partial}{\partial t_3}\{W_1\}\right) + [F_2]\{W_2\} + \{C_4\} = 0 \tag{3.17}$$

The solutions for Eqs. 3.14 to 3.17 are discussed in Subsections 3.3, 3.4, and 3.6.

3.3 The Solutions for $\{W_1\}$ and $\{W_2\}$

The solution for Eq. 3.14 is discussed in Subsection A.7 and is given (disregarding the damped part) by Eq. A.68. As mentioned in Subsection 3.2, this solution contains a secular term. In order to avoid this secular term, it is necessary to assume that $A_1 = 0$ so that Eq. A.68 reduces to

$$\{W_1\} = A\{U\} \tag{3.18}$$

where $\{U\}$ is given by Eq. A.69

$$\{U\} = \begin{Bmatrix} 1 \\ u \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\sqrt{g_1/g_2} \end{Bmatrix} \tag{3.19}$$

and A is a function of t_1, t_2 . . . .

It may be noted that the assumption $A_1 = 0$ does not reduce the generality of the solution. As mentioned in Subsection 3.2, the "versatility" is maintained by introducing the intermediate scales t_1, t_3 . . .

It should be noted also that the solution given by Eq. 3.18 does not depend upon t_o.

However, the complete solution (including the damped terms, see Eq. A.67) depends upon t_o. Thus, t_o is an actual time scale of the phenomenon.

Next, consider the unknown $\{W_2\}$. By combining Eqs. 3.15 and 3.18, one obtains

$$\frac{\partial^2}{\partial t_0^2}\{W_2\} + [G]\frac{\partial}{\partial t_0}\{W_2\} + [F_0]\{W_2\} + \{Z_2\} = 0 \tag{3.20}$$

with

$$\{Z_2\} = [G]\{U\}\frac{\partial A}{\partial t_1} \tag{3.21}$$

It is important to note that the condition for avoiding secular terms

$$\lfloor U^L \rfloor \{Z_2\} = 0 \tag{3.22}$$

is automatically satisfied. Combining Eqs. 3.21 and 3.22 yields

$$\lfloor U^b \rfloor [G] \{U\} \frac{\partial A}{\partial t_1} = 0 \tag{3.23}$$

which is satisfied for any value of $\partial A/\partial t_1$, since by combining Eqs. A.69, A.70, and A.71,

$$\lfloor U^b \rfloor [G] \{U\} = g_1 - g_2 U^2 \equiv 0 \tag{3.24}$$

Thus, the system given by Eq. 3.20 can be solved. The solution is given by (the secular term which appears in Eq. A.68 is dropped, as is done in Eq. 3.18)

$$\{W_2\} = \{U\} B + \{V\} \frac{\partial A}{\partial t_1} \tag{3.25}$$

where

$$\{V\} = [N] [G] \{U\} \tag{3.26}$$

with*

$$[N] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{-1}{\Omega^2_{2*}} \end{bmatrix} \tag{3.27}$$

---

*For many modes, [N] is given by (Subsection 2.3, Ref. 2)

$$[N] = -\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \tilde{F}_0^{-1} \end{array}\right]$$

where $\tilde{F}_o$ is obtained from F_o by eliminating the first row and the first column.

It may be noted that the relation

$$\lfloor U \rfloor [G]\{U\} = 0 \tag{3.28}$$

is highly important in the present analysis. The whole analysis would have little significance if this condition were not valid for more than two modes. Thus, the remainder of this section is devoted to the proof that Eq. 3.28 is valid in general. It may be emphasized that the following discussion is <u>not</u> used in the following sections. It is given only for the sake of generality.

More precisely, it is proved that if the characteristic equation (of the form* Det $[p^2M + pG + F_o] = \Sigma\, c_i p^i = 0$) has a double root $p = 0$, then Eq. 3.28 is satisfied. Because the existence of the double root $p = 0$ implies that the coefficients c_o and c_1 of the characteristic equation are equal to zero; that is

$$\mathrm{Det}\,[F_o] = 0 \tag{3.29}$$

and

$$\frac{\partial}{\partial p}\,\mathrm{Det}\left(p^2[M] + p[G] + [F_o]\right)\Big|_{p=0} = 0 \tag{3.30}$$

The last condition is equivalent to

$$\Sigma\, g_{ki} F_{ki} = 0 \tag{3.31}$$

where F_{ki} is the cofactor of the element f_{ki} of the matrix $[F_o] = [f_{ki}]$. It may be noted that (as is well known), the elements f_{ki} and their cofactors F_{ki} satisfy the relations

*Note that for the sake of generality the matrices M and G are <u>not</u> assumed to be of the diagonal type.

$$\sum_i f_{\ell i} F_{mi} = \text{Det}\,[F_0]\,\delta_{\ell m}$$

$$\sum_i f_{i\ell} F_{im} = \text{Det}\,[F_0]\,\delta_{\ell m} \tag{3.32}$$

or, by employing Eq. 3.29,

$$\sum_i f_{\ell i} F_{mi} = 0$$

$$\sum_i f_{i\ell} F_{im} = 0 \tag{3.33}$$

for any value of ℓ and m. Thus, the components of $\{U\}$ and $\{U^L\}$ which are defined by $\Sigma\, f_{\ell i}\, u_i = 0$ and $\Sigma\, f_{i\ell}\, u_i^L = 0$ can be expressed as (by assuming $F_{m1} \neq 0$ and $F_{1\ell} \neq 0$)

$$u_i = \frac{F_{mi}}{F_{m1}}$$

$$u_k^L = \frac{F_{k\ell}}{F_{1\ell}} \tag{3.34}$$

with arbitrary values of m and ℓ. By choosing m = k and ℓ = 1 (or ℓ = k and m = 1) one obtains

$$\lfloor U^L \rfloor [G]\{U\} = \sum_{k,i} u_k^L g_{ki} u_i = \frac{1}{F_{11}} \sum_{k,i} g_{ki} F_{ki} \tag{3.35}$$

Finally, by employing Eq. 3.31, one obtains the desired Eq. 3.28, which is thus valid for N modes, also.

3.4 The Vector $\{W_3\}$ and the Equation for $A(t_1)$

Combining Eqs. 3.16, 3.18, and 3.25 yields

$$\frac{\partial^2}{\partial t_0^2}\{W_3\} + [G]\frac{\partial}{\partial t_0}\{W_3\} + [F_0]\{W_3\} + \{Z_3\} = 0 \tag{3.36}$$

with

$$\{Z_3\} = \{U\}\frac{\partial^2 A}{\partial t_1^2} + [G]\{U\}\frac{\partial A}{\partial t_2} + [G]\{U\}\frac{\partial B}{\partial t_1}$$
$$+ [G]\{V\}\frac{\partial^2 A}{\partial t_1^2} + [F_2]\{U\}A + \{C_3\} \tag{3.37}$$

where $\{C_3\}$ is given by (see Eq. 3.87)

$$\{C_3\} = \{H_o\} A^3 \tag{3.38}$$

with $\{H_o\}$ given by Eq. 3.88.

The condition for no secular terms in $\{W_3\}$ is

$$\lfloor U^L \rfloor \{Z_3\} = 0 \tag{3.39}$$

with $\lfloor U^L \rfloor$ given by Eq. A.71. Combining Eqs. 3.28, 3.37, 3.38, and 3.39 yields

$$\lfloor U^L \rfloor \left(\{U\} + [G]\{V\} \right) \frac{\partial^2 A}{\partial t_1^2} + \lfloor U^L \rfloor [F_2]\{U\} A + \\ + \lfloor U^L \rfloor \{H_o\} A^3 = 0 \tag{3.40}$$

or

$$\frac{\partial^2 A}{\partial t_1^2} + \beta A + \gamma A^3 = 0 \tag{3.41}$$

where

$$\beta = \frac{1}{\alpha} \lfloor U^L \rfloor [F_2]\{U\} \tag{3.42}$$

$$\gamma = \frac{1}{\alpha} \lfloor U^L \rfloor \{H_o\} \tag{3.43}$$

with

$$\alpha = \lfloor U^L \rfloor \left(\{U\} + [G]\{V\} \right) \tag{3.44}$$

If Eq. 3.41 is satisfied, the solution for Eq. 3.36 is given by (the secular term which appears in Eq. A.68 is dropped, as before, for $\{W_1\}$)

$$\{W_3\} = C\{U\} + \{P_3\} \tag{3.45}$$

where C is a function of t_1, t_2, . . . and $\{P_3\}$ is given by

$$\begin{aligned} \{P_3\} &= [N]\{Z_3\} \\ &= [P_\alpha]\frac{\partial^2 A}{\partial t_1^2} + [P_\beta]A + [P_\gamma]A^3 + \{V\}\left(\frac{\partial A}{\partial t_2} + \frac{\partial B}{\partial t_1}\right) \end{aligned} \tag{3.46}$$

with [N] given by Eq. 3.27, {V} given by Eq. 3.26, and

$$\begin{aligned} \{P_\alpha\} &= [N]\left(\{U\} + [G]\{V\}\right) \\ \{P_\beta\} &= [N]\,[F_2]\{U\} \\ \{P_\gamma\} &= [N]\{H_0\} \end{aligned} \tag{3.47}$$

Note that $\{P_\alpha\}$, $\{P_\beta\}$ and $\{P_\gamma\}$ are originated by the same terms which yield α, β and γ, respectively. The solution for Eq. 3.41 is discussed in Subsection 3.5.

3.5 The Function $A(t_1)$

Next, the solution $A(t_1)$ for Eq. 3.41 is discussed. Note first that Eq. 3.41 can be integrated once to give

$$\frac{1}{2}\left(\frac{\partial A}{\partial t_1}\right)^2 + \beta\frac{A^2}{2} + \gamma\frac{A^4}{4} = \mathcal{E}(t_2) \tag{3.48}$$

where $\mathcal{E}$ is a constant of integration (actually, a function of t_2), which represents the energy $\mathcal{E}$ of the mass-nonlinear-spring system described by Eq. 3.41. Thus A is obtained by considering the inverse function of the integral

$$t_1 + \tau = \int^{A} \frac{da}{\sqrt{2\mathcal{E} - \beta a^2 - \gamma a^4/2}} \tag{3.49}$$

The integral in Eq. 3.49 is an elliptic integral. The properties of the elliptic integral are given in Refs. 4, 5, 6, and 7. The inverse function of the elliptic integrals are the elliptic functions, among which the most useful ones for the present study are the Jacobian elliptic functions. Some properties of the elliptic functions are given in Subsection 3.10.

In the following, only the case

$$\gamma > 0 \tag{3.50}$$

(which is the most important in the practical application) is discussed. The case $\gamma < 0$ can be treated in a very similar way. As it appears clear from Eq. 3.41, the case $\gamma > 0$ corresponds to the case of a "hard-spring-nonlinear term". Note, first, that the radicand of Eq. 3.49 is equal to zero for

$$A^2 = \frac{-\beta}{\gamma}\left(1 \pm \sqrt{1 + \frac{4\mathcal{E}\gamma}{\beta^2}}\right) = \begin{cases} A^2_{(+)} \\ A^2_{(-)} \end{cases} \tag{3.51}$$

The function $A(t_1)$ depends essentially upon the sign of $A^2_{(+)}$ and $A^2_{(-)}$. Only two cases are of interest here. All the other possibilities do not correspond to periodic solutions.

3.5.1 Unbuckled Case

Both $A^2_{(+)}$ and $A^2_{(-)}$ are positive. This implies that (since $\gamma > 0$, according to Eq. 3.50)

$$\beta < 0 \tag{3.52a}$$

and

$$-\frac{\beta^2}{4\gamma} < \mathcal{E} < 0 \tag{3.52b}$$

and the solution is of the form (Ref. 4, Eq. 17.4.52)

$$A = A_o \, cn \left[\omega (t_1 + t_{1,o}), K_u \right] \tag{3.53}$$

The dependence of A_o, ω, and K_u upon β, γ, and $\mathcal{E}$ is obtained by combining Eqs. 3.48 and 3.53 and by using Eqs. 3.134 and 1.136*

$$\mathcal{E} = \frac{\beta}{2} A_o^2 + \frac{\gamma}{4} A_o^4 \tag{3.54}$$

$$\omega_u^2 = \beta + \gamma A_o^2 \tag{3.55}$$

$$K_u^2 = \frac{1}{2} \frac{\gamma A_o^2}{\beta + \gamma A_o^2} \tag{3.56}$$

This solution will be referred to as the "unbuckled solution" since it corresponds to a plate vibrating about a flat position (see Fig. 3b).

3.5.2 Buckled Plate

One of the values of $A^2_{(+)}$ and $A^2_{(-)}$ is positive and the other is negative. This implies that

---

*By using Eqs. 3.134 and 3.136

$$A = A_o C = A_o \sqrt{1 - S^2} \qquad \frac{dA}{dt_1} = -A_o \omega_u S d = -A_o \omega_u S \sqrt{1 - K_u^2 S^2}$$

and combining with Eq. 3.48

$$\begin{aligned} \frac{1}{2}\left(\frac{dA}{dt_1}\right)^2 + \beta\frac{A^2}{2} + \gamma\frac{A^4}{4} - \mathcal{E} &= \frac{1}{2} A_o^2 \omega_u^2 S^2 (1 - K_u^2 S^2) + \frac{1}{2} \beta A_o^2 (1 - S^2) \\ &+ \frac{1}{4} \gamma A_o^4 (1 - 2S^2 + S^4) - \mathcal{E} = \left(\frac{1}{2}\beta A_o^2 + \frac{1}{4}\gamma A_o^4 - \mathcal{E}\right) \\ &+ S^2 \left(\frac{1}{2} A_o^2 \omega_u^2 - \frac{1}{2}\beta A_o^2 - \frac{1}{2}\gamma A_o^4\right) + S^4 \left(-\frac{1}{2} A_o^2 \omega_u^2 K_u^2 + \frac{\gamma}{4} A_o^4\right) = 0 \end{aligned}$$

which is equivalent to Eqs. 3.54, 3.55, and 3.56.

$$\mathcal{E}_B > 0 \tag{3.57}$$

and the solution is of the form (Ref. 4, Eq. 17.4.44)

$$A = A_o \, dn \left[\omega (t + t_{1,0}) , K_B \right] \tag{3.58}$$

The dependence of A_o, ω and K upon β, γ and ε is obtained by combining Eqs. 3.48 and 3.58 and by using Eqs. 3.134, 3.135, and 3.136*

$$\mathcal{E} = \frac{1}{2} \beta A_o^2 + \frac{1}{4} \gamma A_o^4 \tag{3.59}$$

$$\omega_B^2 = \frac{1}{2} \gamma A_o^2 \tag{3.60}$$

$$K_B^2 = \frac{2(\beta + \gamma A_o^2)}{\gamma A_o^2} \tag{3.61}$$

---

*By using Eqs. 3.134, 3.135, and 3.136

$$A = A_o d = A_o \sqrt{1 - K_B^2 S^2} , \quad \frac{dA}{dt_1} = A_o \omega_B (-K_B^2 S C) = -A_o \omega_B K_B^2 S \sqrt{1 - S^2}$$

and combining with Eq. 3.48

$$\left(\frac{dA}{dt}\right)^2 - \beta A^2 + \gamma \frac{A^4}{2} - 2\mathcal{E} = A_o^2 \omega_B^2 K_B^4 S^2 (1 - S^2) + \beta A_o^2 (1 - K_B^2 S^2)$$
$$+ \frac{\gamma}{2} A_o^4 (1 - 2K_B^2 S^2 + K_B^4 S^4) - 2\mathcal{E} = (\beta A_o^2 + \frac{\gamma}{2} A_o^4 - 2\mathcal{E})$$
$$+ S^2 [A_o^2 \omega_B^2 K_B^4 - \beta A_o^2 K_B^2 - \gamma A_o^4 K_B^2] + S^4 (-A_o^2 \omega_B^2 K_B^4 + \frac{\gamma}{2} A_o^4 K_B^4) = 0$$

which is equivalent to Eqs. 3.59, 3.60, and 3.61.

This solution will be referred to as the "buckled solution" since it corresponds to a plate fluttering around a buckled position (see Fig. 3c).

3.6 The Vector $\{W_4\}$ and the Equation for $B(t_1)$

Next consider the vector $\{W_4\}$. Combining Eqs. 3.17 with the expressions for $\{W_1\}$ (Eq. 3.18), $\{W_2\}$ (Eq. 3.25), $\{W_3\}$ (Eq. 3.45) and $\{C_4\}$ (Eq. 3.91) yields

$$\frac{\partial^2\{W_4\}}{\partial t_0^2} + [G]\frac{\partial}{\partial t_0}\{W_4\} + [F_0]\{W_4\} + [Z_4] = 0 \tag{3.62}$$

with

$$\begin{aligned}
\{Z_4\} &= 2\frac{\partial^2}{\partial t_1 \partial t_2}\{W_1\} + \frac{\partial^2}{\partial t_1^2}\{W_2\} + [F_2]\{W_2\} + \{C_4\} + \\
&+ [G]\left(\frac{\partial}{\partial t_3}\{W_1\} + \frac{\partial}{\partial t_2}\{W_2\} + \frac{\partial}{\partial t_1}\{W_3\}\right) \\
&= 2\{U\}\frac{\partial^2 A}{\partial t_1 \partial t_2} + \{U\}\frac{\partial^2 B}{\partial t_1^2} + \{V\}\frac{\partial^3 A}{\partial t_1^3} + \\
&+ [G]\left[\frac{\partial A}{\partial t_3}\{U\} + \frac{\partial B}{\partial t_2}\{U\} + \{V\}\frac{\partial^2 A}{\partial t_1 \partial t_2} + \right. \\
&+ \{U\}\frac{\partial C}{\partial t_1} + \{P_\alpha\}\frac{\partial^3 A}{\partial t_1^3} + \{P_\beta\}\frac{\partial A}{\partial t_1} + \\
&\left. + \{P_\gamma\}\frac{\partial}{\partial t_1}(A^3) + \{V\}\left(\frac{\partial^2 A}{\partial t_1 \partial t_2} + \frac{\partial^2 B}{\partial t_1^2}\right)\right] \\
&+ [F_2]\{U\}B + [F_2]\{V\}\frac{\partial A}{\partial t_1} + 3A^2B\{H_0\} + \\
&+ A^2\{H_1\}\frac{\partial A}{\partial t_1}
\end{aligned} \tag{3.63}$$

where $\{H_o\}$ and $\{H_1\}$ are given by Eqs. 3.88 and 3.92.

The condition for no secular terms in $\{W_4\}$ is

$$\lfloor U^L \rfloor \{Z_4\} = 0 \tag{3.64}$$

with $[U^L]$ given by Eq. A.71. Combining Eqs. 3.28, 3.63, and 3.64 yields

$$\begin{aligned}
&\lfloor U^L \rfloor \left(\{U\} + [G]\{V\}\right)\frac{\partial^2 B}{\partial t_1^2} + \lfloor U^L \rfloor [F_2]\{U\}B \\
&+ 3A^2 B \lfloor U^L \rfloor \{H_o\} + \lfloor U^L \rfloor \left(2\{U\} + [G]\{V\} \times 2\right)\frac{\partial^2 A}{\partial t_1 \partial t_2} \\
&+ \lfloor U^L \rfloor \left([G]\{P_\alpha\} + \{V\}\right)\frac{\partial^3 A}{\partial t_1^3} \\
&+ \lfloor U^L \rfloor \left([G]\{P_\beta\} + \{F_2\}\{V\}\right)\frac{\partial A}{\partial t_1} \\
&+ \lfloor U^L \rfloor \left(3[G]\{P_\gamma\} + \{H_1\}\right)A^2\frac{\partial A}{\partial t_1} = 0
\end{aligned} \tag{3.65}$$

It may be noted that the terms with $\partial A/\partial t_3$, $\partial B/\partial t_2$, and $\partial C/\partial t_1$ disappear because of Eq. 3.28. Equation 3.65 can be rewritten as

$$\begin{aligned}
&\frac{\partial^2 B}{\partial t_1^2} + \beta B + 3\gamma A^2 B + 2\frac{\partial^2 A}{\partial t_1 \partial t_2} \\
&\qquad + \frac{\alpha_1}{\alpha}\left(\frac{\partial^3 A}{\partial t_1^3} + \beta_1 \frac{\partial A}{\partial t_1} + 3\gamma A^2 \frac{\partial A}{\partial t_1}\right) = 0
\end{aligned} \tag{3.66}$$

where (in similarity with Eqs. 3.42, 3.43, and 3.44)

$$\begin{aligned}
\alpha_1 &= \lfloor U^L \rfloor \left(\{V\} + [G]\{P_\alpha\}\right) \\
\beta_1 &= \frac{1}{\alpha_1}\lfloor U^L \rfloor \left([F_2]\{V\} + [G]\{P_\beta\}\right) \\
\gamma_1 &= \frac{1}{\alpha_1}\lfloor U^L \rfloor \left(\frac{1}{3}\{H_1\} + [G]\{P_\gamma\}\right)
\end{aligned} \tag{3.67}$$

and α, β and γ are given by Eqs. 3.44, 3.43, and 3.42.

Finally, if Eq. 3.66 is satisfied, the solution for Eq. 3.62 is given by (the secular term which appears in Eq. A.68 is dropped, as before for $\{w_1\}$).

$$\{W_4\} = D\{U\} + \{P_4\} \tag{3.68}$$

where

$$\{P_4\} = [N]\{Z_4\} \tag{3.69}$$

with [N] given by Eq. 3.27.

The solution for Eq. 3.66 is discussed in Subsection 3.7.

3.7 The Function $B(t_1)$

In the preceding subsection, it was shown that in order to avoid secular terms in the solution for $\{w_4\}$, the condition expressed by Eq. 3.66 must be satisfied. This equation can be rewritten as

$$\frac{\partial^2 B}{\partial t_1^2} + \beta B + 3\gamma A^2 B + \left(\frac{\partial A}{\partial t_1}\right)^{-1} \delta = 0 \tag{3.70}$$

with

$$\delta = \left[2\frac{\partial^2 A}{\partial t_2 \partial t_1} + \frac{\alpha_1}{\alpha}\left(\frac{\partial^3 A}{\partial t_1^3} + \beta_1 \frac{\partial A}{\partial t_1} + 3\gamma A^2 \frac{\partial A}{\partial t_1} \right)\right]\frac{\partial A}{\partial t_1}$$

$$= \frac{\partial}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2 + \tilde{\beta}\left(\frac{\partial A}{\partial t_1}\right)^2 + 3\tilde{\gamma} A^2 \left(\frac{\partial A}{\partial t_1}\right)^2 \tag{3.71}$$

with

$$\tilde{\beta} = \frac{\alpha_1}{\alpha}(\beta_1 - \beta) \tag{3.72}$$

$$\tilde{\gamma} = \frac{\alpha_1}{\alpha}(\gamma_1 - \gamma) \tag{3.73}$$

The last expression for δ is obtained by using the derivative of Eq. 3.41

$$\frac{\partial^3 A}{\partial t_1^3} + \beta \frac{\partial A}{\partial t_1} + 3\gamma A^2 \frac{\partial A}{\partial t_1} = 0 \tag{3.74}$$

In order to solve Eq. 3.70, set

$$B = \frac{\partial A}{\partial t_1} b \tag{3.75}$$

Combining Eqs. 3.70 and 3.74 yields

$$A'''b + A'b'' + 2A''b' + \beta A'b + 3\gamma A^2 A'b + \frac{1}{A'}\delta = 0 \tag{3.76}$$

with $(\ )' = \frac{\partial}{\partial t_1}(\ )$, and using Eq. 3.74

$$2A''b' + A'b'' + \frac{1}{A'}\delta = 0 \tag{3.77}$$

which is equivalent to

$$(b'A'^2)' + \delta = 0 \tag{3.78}$$

or

$$b = -\int \frac{1}{\left(\frac{\partial A}{\partial t_1}\right)^2} I\, dt_1 + b_0 \tag{3.79}$$

with

$$I = \int \delta\, dt_1 + \tilde{C} \tag{3.80}$$

The integral I is discussed in Subsection 3.9 and is given by Eq. 3.117. The explicit expression for the function b is given by Eq. 3.123.

It can be seen that if $\partial \mathcal{E}/\partial t_2$ is used to avoid secular terms of the type $t_1 Q$* then it is not possible to eliminate the variable t_1 in the equation for $\tau(t_2)$. In other words, if $\partial \mathcal{E}/\partial t_2 \neq 0$, it is impossible to find a function τ independent of t_1**.

On the other hand, the equation for $\mathcal{E}(t_2)$ is so complicated that it is not of particular interest here. From the viewpoint of the application, it is more interesting to consider the steady-state problem (limit-cycle solution) for which

$$\frac{\partial \mathcal{E}}{\partial t_2} = 0 \tag{3.81}$$

Then the integral I, given by Eq. 3.117 reduces to (for $\tilde{c} = 0$)

$$I = \frac{\partial \tau}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2 + \tilde{C}_0 t_1 + \tilde{C}_2 I^{(2)} + \tilde{C}_4 A \frac{\partial A}{\partial t_1} + \tilde{C}_6 A^3 \frac{\partial A}{\partial t_1} \tag{3.82}$$

Equation 3.82 contains secular terms of the form $C_{S.T.} t_1$ where the expression for $C_{S.T.}$ is different for the unbuckled and the buckled cases, as follows For the unbuckled case (that is, for A given by Eq. 3.53 and $I^{(2)}$ given by Eq. 3.112):***

$$C_{S.T.} = \tilde{C}_0 + \tilde{C}_2 A_0^2 \left[\frac{\tilde{E}}{\tilde{k}} - (1 - K^2)\right]\frac{1}{K^2} \tag{3.83}$$

whereas for the buckled case (that is, for A given by Eq. 3.58 and $I^{(2)}$ given by Eq. 3.113):

$$C_{S.T.} = \tilde{C}_0 + \tilde{C}_2 A_0^2 \frac{\tilde{E}}{\tilde{k}} \tag{3.84}$$

In Eqs. 3.83 and 3.84, $\tilde{E}$ and $\tilde{K}$ are given by Eq. 3.138.

---

*By setting $\partial \mathcal{E}/\partial t_2 + c_o = 0$, the terms of type $t_1 Q$ are eliminated from Eq. 3.123, but the terms of the type $Q\, dt_1$ are retained.

**For a better understanding of this question, see Sections 4 and 5.

***Equations 3.114 and 3.137 are used here.

The condition for no secular terms

$$C_{S.T.} = 0 \tag{3.85}$$

yields an equation for the amplitude of the limit cycle A_o. Note that this equation is of transcendental form since K (given by Eqs. 3.56 or 3.61) and thus $\tilde{E}$ and $\tilde{K}$ depend upon A_o. Once the secular terms have been eliminated from I, Eq. 3.79 can be used to evaluate b; the secular term in b can be avoided by assuming a suitable constant value $\tau^{(1)}$ for $\partial\tau/\partial t_2$ (actually $\tau^{(1)}$ is a function of t_3 . . .); $\tau^{(1)}$ gives the dependence of the period upon ε.

3.8 The Nonlinear Terms

In this subsection, explicit expressions for $\{C_3\}$ and $\{C_4\}$ (defined by Eqs. 3.12 and 3.13) are derived. Consider, first, the vector $\{C_3\}$. Note that Eq. 3.18 can be rewritten as

$$\{W_1\} = \begin{Bmatrix} W_{11} \\ W_{21} \end{Bmatrix} = A \begin{Bmatrix} 1 \\ u \end{Bmatrix} \tag{3.86}$$

Combining Eqs. 3.12 and 3.86 yields

$$\{C_3\} = A^3 \begin{Bmatrix} C_{11} + C_{12}u^2 \\ C_{21}u + C_{22}u^3 \end{Bmatrix} = A^3\{H_o\} \tag{3.87}$$

with

$$\{H_o\} = \begin{Bmatrix} C_{11} + C_{12}u^2 \\ C_{21}u + C_{22}u^3 \end{Bmatrix} \tag{3.88}$$

Next, consider the vector $\{C_4\}$. Note that Eq. 3.25 can be rewritten as

$$\{W_2\} = \begin{Bmatrix} W_{12} \\ W_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ u \end{Bmatrix} B + \begin{Bmatrix} 0 \\ v \end{Bmatrix} \frac{\partial A}{\partial t_1} \tag{3.89}$$

with

$$v = -\frac{1}{\Omega_{2*}^2} g_2 u \tag{3.90}$$

Combining Eqs. 3.13, 3.86, and 3.89 yields

$$\begin{aligned} \{C_4\} &= 3A^2B \begin{Bmatrix} C_{11} + C_{12}u^2 \\ C_{21}u + C_{22}u^3 \end{Bmatrix} \\ &\quad + A^2\frac{\partial A}{\partial t_1} \begin{Bmatrix} 2C_{12}uv \\ C_{21}v + 3C_{22}vu^2 \end{Bmatrix} \\ &= 3A^2B\{H_0\} + A^2\frac{\partial A}{\partial t_1}\{H_1\} \end{aligned} \tag{3.91}$$

with $\{H_o\}$ given by Eq. 3.88 and

$$\{H_1\} = \begin{Bmatrix} 2C_{12}uv \\ C_{21}v + 3C_{22}vu^2 \end{Bmatrix} = \frac{\partial}{\partial u}\{H_0\}v \tag{3.92}$$

3.9 Mathematical Elaborations

Consider Eqs. 3.79 and 3.80

$$b = -\int \frac{1}{\left(\frac{\partial A}{\partial t_1}\right)^2} I\, dt_1 + b_0 \tag{3.93}$$

with

$$I = \int \delta \, dt_1 + \tilde{C} \tag{3.94}$$

where

$$\delta = \frac{\partial}{\partial t_2} \left(\frac{\partial A}{\partial t_1} \right)^2 + (\tilde{\beta} + 3\tilde{\gamma} A^2) \left(\frac{\partial A}{\partial t_1} \right)^2 \tag{3.95}$$

In this subsection, the explicit expression for b is obtained. The secular terms are included in the analysis. The discussion of the secular terms is given in Subsection 3.7. Equation 3.94 can be rewritten as

$$I = I_1 + I_2 + \tilde{C} \tag{3.96}$$

with

$$I_1 = \int \frac{\partial}{\partial t_2} \left(\frac{\partial A}{\partial t_1} \right)^2 dt_1 \tag{3.97}$$

and

$$\begin{aligned} I_2 &= \int (\tilde{\beta} + 3\tilde{\gamma} A^2) \left(\frac{\partial A}{\partial t_1} \right)^2 dt_1 \\ &= \int (\tilde{\beta} + 3\tilde{\gamma} A^2)(2\mathcal{E} - \beta A^2 - \frac{\gamma}{2} A^4) dt_1 \\ &= \int (C_0 + C_2 A^2 + C_4 A^4 + C_6 A^6) dt_1 \\ &= C_0 I^{(0)} + C_2 I^{(2)} + C_4 I^{(4)} + C_6 I^{(6)} . \end{aligned} \tag{3.98}$$

with

$$C_0 = 2\tilde{\beta}\mathcal{E} , \quad C_2 = -\tilde{\beta}\beta + 6\tilde{\gamma}\mathcal{E}$$
$$C_4 = -\frac{1}{2}\tilde{\beta}\gamma - 3\tilde{\gamma}\beta , \quad C_6 = -\frac{3}{2}\tilde{\gamma}\gamma \tag{3.99}$$

and

$$I^{(k)} = \int A^k dt_1 = \int A^k \frac{1}{\frac{\partial A}{\partial t_1}} dA \tag{3.100}$$

3.9.1 The Integral I_1

Consider first, the integral I_1, given by Eq. 3.97. Note that differentiating Eq. 3.48 with respect to t_2 yields (by using Eq. 3.41)

$$\begin{aligned} \frac{\partial}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2 &= 2\,\frac{\partial \mathcal{E}}{\partial t_2} - 2\,(\beta A + \gamma A^3)\,\frac{\partial A}{\partial t_2} \\ &= 2\left(\frac{\partial \mathcal{E}}{\partial t_2} + \frac{\partial^2 A}{\partial t_1^2}\,\frac{\partial A}{\partial t_2}\right) \end{aligned} \tag{3.101}$$

In order to find $\partial A/\partial t_2$, it is convenient to follow a process discovered by Hermite (Ref. 6, p. 245): differentiating Eq. 3.49 by having in mind that only A, τ, and $\mathcal{E}$ depend upon t_2(t_1 and t_2 are independent variables) yields

$$\frac{\partial \tau}{\partial t_2} = \frac{1}{\sqrt{2\mathcal{E} - \beta A^2 - \frac{\gamma}{2} A^4}}\,\frac{\partial A}{\partial t_2} - \frac{\partial \mathcal{E}}{\partial t_2}\int^A \frac{da}{(2\mathcal{E} - \beta a^2 - \frac{\gamma}{2} a^4)^{3/2}} \tag{3.102}$$

Equation 3.102 is equivalent to

$$\frac{\partial A}{\partial t_2} = \frac{\partial A}{\partial t_1}\left(\frac{\partial \tau}{\partial t_2} + \frac{\partial \mathcal{E}}{\partial t_2}\,Q\right) \tag{3.103}$$

where

$$Q(t_1) = \int \frac{da}{(2\mathcal{E} - \beta a^2 - \frac{\gamma}{2} a^4)^{3/2}} = \int_0^{t_1}\left(\frac{\partial A}{\partial t_1}\right)^{-2} dt_1 \tag{3.104}$$

Combining Eqs. 3.101 and 3.103, one obtains

$$\frac{\partial}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2 = 2\,\frac{\partial \mathcal{E}}{\partial t_2} + 2\,\frac{\partial^2 A}{\partial t_1^2}\,\frac{\partial A}{\partial t_1}\left(\frac{\partial \tau}{\partial t_2} + \frac{\partial \mathcal{E}}{\partial t_2}Q\right)$$

$$= \frac{\partial \mathcal{E}}{\partial t_2}\left\{2 + \frac{\partial}{\partial t_1}\left(\frac{\partial A}{\partial t_1}\right)^2 Q\right\} + \frac{\partial}{\partial t_1}\left(\frac{\partial A}{\partial t_1}\right)^2 \frac{\partial \tau}{\partial t_2} \qquad (3.105)$$

Thus, combining Eqs. 3.97 and 3.105 yields*

$$I_1 = \frac{\partial \mathcal{E}}{\partial t_2}\int\left[2 + \frac{\partial}{\partial t_1}\left(\frac{\partial A}{\partial t_1}\right)^2 Q\right]dt_1 + \frac{\partial \tau}{\partial t_2}\int \frac{\partial}{\partial t_1}\left(\frac{\partial A}{\partial t_1}\right)^2 dt_1$$

$$= \frac{\partial \mathcal{E}}{\partial t_2}\left[t_1 + \left(\frac{\partial A}{\partial t_1}\right)^2 Q\right] + \frac{\partial \tau}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2$$

$$= \frac{\partial \mathcal{E}}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2 \frac{\partial}{\partial t_1}(t_1 Q) + \frac{\partial \tau}{\partial t_2}\left(\frac{\partial A}{\partial t_1}\right)^2 \qquad (3.106)$$

3.9.2 The Integral I_2

Consider Eq. 3.98 which involves the evaluation of the integrals $I^{(o)}$, $I^{(2)}$, $I^{(4)}$, and $I^{(6)}$. Note, that according to Eq. 3.100

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*Note, that by integrating by parts

$$\int \frac{\partial}{\partial t_1}\left(\frac{\partial A}{\partial t_1}\right)^2 Q\,dt = \left(\frac{\partial A}{\partial t_1}\right)^2 Q - \int\left(\frac{\partial A}{\partial t_1}\right)^2 \frac{\partial Q}{\partial t_1}\,dt_1$$

$$= \left(\frac{\partial A}{\partial t_1}\right)^2 Q - t_1$$

since

$$\frac{\partial Q}{\partial t_1} = \left(\frac{\partial A}{\partial t_1}\right)^{-2}$$

$$I^{(0)} = \int dt_1 = t_1 \tag{3.107}$$

Furthermore*

$$I^{(4)} = -\frac{4\beta}{3\gamma} I^{(2)} + \frac{4\mathcal{E}}{3\gamma} I^{(0)} - \frac{2}{3\gamma}\frac{\partial A}{\partial t_1} A \tag{3.108}$$

and

$$\begin{aligned} I^{(6)} &= -\frac{8\beta}{5\gamma} I^{(4)} + \frac{12\mathcal{E}}{5\gamma} I^{(2)} - \frac{2}{5\gamma}\frac{\partial A}{\partial t_1} A^3 \\ &= \left(\frac{32\beta^2}{15\gamma^2} + \frac{12\mathcal{E}}{5\gamma}\right) I^{(2)} - \frac{32\beta\mathcal{E}}{15\gamma^2} I^{(0)} + \frac{16\beta}{15\gamma^2}\frac{\partial A}{\partial t_1} A - \frac{2}{5\gamma}\frac{\partial A}{\partial t_1} A^3 \end{aligned} \tag{3.109}$$

Thus, combining Eqs. 3.98, 3.107, 3.108, and 3.109 yields

$$\begin{aligned} I_2 &= \left(C_0 + C_4\frac{4\mathcal{E}}{3\gamma} - C_6\frac{32\beta\mathcal{E}}{15\gamma^2}\right)t_1 + \Big[C_2 - C_4\frac{4\beta}{3\gamma} \\ &\quad + C_6\left(\frac{32\beta^2}{15\gamma^2} + \frac{12\mathcal{E}}{5\gamma}\right)\Big] I^{(2)} - \left(C_4\frac{2}{3\gamma} - C_6\frac{16\beta}{15\gamma^2}\right)\frac{\partial A}{\partial t_1} A \\ &\quad - C_6\frac{2}{5\gamma}\frac{\partial A}{\partial t_1} A^3 = \tilde{C}_0 t_1 + \tilde{C}_2 I^{(2)} + \tilde{C}_4\frac{\partial A}{\partial t_1} A + \tilde{C}_6\frac{\partial A}{\partial t_1} A^3 \end{aligned} \tag{3.110}$$

---

*Following a procedure used in Ref. 5, pp. 605-607, note that

$$\begin{aligned} \frac{\partial}{\partial A}\left(\frac{\partial A}{\partial t_1} A^n\right) &= \frac{\partial}{\partial A}\left(\sqrt{2\mathcal{E} - \beta A^2 - \gamma\frac{A^4}{2}}\, A^n\right) = \Bigg(\frac{1}{2}\,\frac{A^n}{\sqrt{2\mathcal{E} - \beta A^2 - \gamma\frac{A^4}{2}}} \\ &\quad + \sqrt{2\mathcal{E} - \beta A^2 - \gamma\frac{A^4}{2}}\; n A^{n-1}\Bigg) = \left[2\mathcal{E} n A^{n-1} - \beta(n+1)A^{n+1} - \frac{\gamma}{2}(n+2)A^{n+3}\right]\frac{1}{\frac{\partial A}{\partial t_1}} \end{aligned}$$

which is equivalent to

$$\frac{\partial A}{\partial t_1} A^n = 2\mathcal{E} n\, I^{(n-1)} - \beta(n+1)\, I^{(n+1)} - \frac{\gamma}{2}(n+2)\, I^{n+3}$$

from which Eqs. 3.108 and 3.109 are obtained for n = 1 and n = 3, respectively.

with

$$\tilde{C}_0 = C_0 + C_4 \frac{4\mathcal{E}}{3\gamma} - C_6 \frac{32\beta\mathcal{E}}{15\gamma^2}$$

$$\tilde{C}_2 = C_2 - C_4 \frac{4\mathcal{E}}{3\gamma} + C_6 \left(\frac{32\beta^2}{15\gamma^2} + \frac{12\mathcal{E}}{5\gamma} \right)$$

$$\tilde{C}_4 = -C_4 \frac{2}{3\gamma} + C_6 \frac{16\beta}{15\gamma^2}$$

$$\tilde{C}_6 = - C_6 \frac{2}{5\gamma} \tag{3.111}$$

The integral $I^{(2)}$ cannot be expressed in terms of the Jacobian elliptic function. For the unbuckled case (A given by Eq. 3.53), $I^{(2)}$ is given by (Ref. 6, p.235)

$$I^{(2)} = A_0^2 \int cn^2(\omega, t_1)\, dt_1 = \frac{A_0^2}{\omega} C_n(\omega t_1) \tag{3.112}$$

whereas for the buckled case (A given by Eq. 3.58)

$$I^{(2)} = A_0^2 \int dn^2(\omega, t)\, dt_1 = \frac{A_0^2}{\omega} D_n\, \omega t_1 \tag{3.113}$$

with (see Eq. 3.141)

$$C_n(u) = \left\{ E(u) - (1 - K^2) u \right\} / K^2$$
$$D_n(u) = E(u) \tag{3.114}$$

where E(u) is the Legendre first elliptic integral. Note that

$$E(u) = \frac{\tilde{E}}{\tilde{K}} u + \textit{periodic function} \tag{3.115}$$

as is shown by Eq. 3.137.

3.9.3 The Function b

Consider the function b as defined by Eq. 3.93

$$b = \int \left(\frac{\partial A}{\partial t_1}\right)^{-2} I \, dt_1 + b_0 \tag{3.116}$$

where I is obtained by combining Eqs. 3.96, 3.106, and 3.110

$$I = \left[\frac{\partial \mathcal{E}}{\partial t_2} \frac{\partial}{\partial t_1}(t_1 Q) + \frac{\partial \tau}{\partial t_2} \right] \left(\frac{\partial A}{\partial t_1}\right)^2 + \tilde{C}_0 t_1 + \tilde{C}_2 I^{(2)}$$

$$+ \tilde{C}_4 \frac{\partial A}{\partial t_1} A + \tilde{C}_6 \frac{\partial A}{\partial t_1} A^3 + \tilde{C} \tag{3.117}$$

Combining Eqs. 3.116 and 3.117

$$b = \frac{\partial \mathcal{E}}{\partial t_2} t_1 Q + \frac{\partial \tau}{\partial t_2} t_1 + \tilde{C}_0 \int \left(\frac{\partial A}{\partial t_1}\right)^{-2} t_1 \, dt_1 + \tilde{C}_2 \int \left(\frac{\partial A}{\partial t_1}\right)^{-2} I^{(2)} dt_1$$

$$+ \tilde{C}_4 \int A \left(\frac{\partial A}{\partial t_1}\right)^{-1} dt_1 + \tilde{C}_6 \int A^3 \left(\frac{\partial A}{\partial t_1}\right)^{-1} dt_1 + \tilde{C} Q + b_0 \tag{3.118}$$

Note that by integrating by parts

$$\int \left(\frac{\partial A}{\partial t_1}\right)^{-2} t_1 \, dt_1 = t_1 Q - \int Q \, dt_1 \tag{3.119}$$

Furthermore,*

$$J^{(1)} = \int A\left(\frac{\partial A}{\partial t_1}\right)^{-1} dt = \int A\left(\frac{\partial A}{\partial t_1}\right)^{-2} dA$$

$$= \frac{1}{\sqrt{\beta^2 - 4\gamma\mathcal{E}}} \ln\left|\frac{A^2 - A_+^2}{A^2 - A_-^2}\right| \tag{3.120}$$

with

$$A_\pm^2 = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma\mathcal{E}}}{\gamma} \tag{3.121}$$

Finally,**

$$J^{(3)} = \int A^3\left(\frac{\partial A}{\partial t_1}\right)^{-1} dt = \int A^3\left(\frac{\partial A}{\partial t_1}\right)^{-2} dA$$

$$= -\frac{\beta}{\gamma} J^{(1)} - \frac{1}{\gamma} \ln \frac{\partial A}{\partial t_1} \tag{3.122}$$

Thus, combining Eqs. 3.118, 3.120, and 3.122 yields

$$b = \frac{\partial \mathcal{E}}{\partial t_2} t_1 Q + \frac{\partial \tau}{\partial t_2} t_1 + \tilde{C}_0 \left(t_1 Q - \int Q\, dt_1\right)$$

$$+ \tilde{C}_2 \int I^{(2)}\left(\frac{\partial A}{\partial t_1}\right)^{-2} dt_1 + \left(\tilde{C}_4 - \frac{\beta}{\gamma}\tilde{C}_6\right) J^{(1)} - \tilde{C}_6 \frac{1}{\gamma} \ln \frac{\partial A}{\partial t_1}$$

$$+ \tilde{C} Q + b_0 \tag{3.123}$$

---

*By setting $A^2 = \xi$, one obtains the elementary integral

$$J^{(1)} = \frac{1}{2}\int \left(2\mathcal{E} - \beta\xi - \frac{\gamma}{2}\xi^2\right)^{-1} d\xi$$

**Note that

$$\frac{\partial}{\partial A} \ln\left(\frac{\partial A}{\partial t_1}\right)^2 = \left(\frac{\partial A}{\partial t_1}\right)^{-2}\left(-2\beta A - 2\gamma A^3\right)$$

or

$$\frac{1}{2}\ln\left(\frac{\partial A}{\partial t_1}\right)^2 = -\beta J^{(1)} - \gamma J^{(3)}$$

3.9.4 The Function Q

Consider the function Q defined by Eq. 3.104

$$Q = \int \left(\frac{\partial A}{\partial t_1}\right)^{-2} dt_1 \tag{3.124}$$

where (using Eq. 3.136)

$$\frac{\partial A}{\partial t_1} = -A_0 \, sn(\omega t_1) \, dn(\omega t_1) \tag{3.125}$$

for the unbuckled case (A given by Eq. 3.53) and

$$\frac{\partial A}{\partial t_1} = -A_0 K^2 cn(\omega t_1) \, sn(\omega t_1) \tag{3.126}$$

for the buckled case (A given by Eq. 3.58). Note that* (see Eqs. 3.134, 3.135, and 3.139)

$$\frac{1}{sn^2 dn^2} = ns^2 + K^2 nd^2 \tag{3.127}$$

and

$$\frac{1}{sn^2 cn^2} = ns^2 + nc^2 \tag{3.128}$$

Thus for the unbuckled case

$$Q = \frac{1}{A_0^2} \int \frac{1}{sn^2 dn^2} dt_1 = \frac{1}{A_0^2 \omega^2} \int (ns^2 + K^2 nd^2) du$$

$$= \frac{1}{A_0^2 \omega} (Ns\, u + K^2 Nd\, u) \qquad \textbf{(unbuckled)} \tag{3.129}$$

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*The general procedure is given in Ref. 6 where use of the general Glaisdel function is made.

and for the buckled case

$$Q = \frac{1}{A_0^2 K^2}\int \frac{1}{sn^2\, cn^2}\,dt_1 = \frac{1}{A_0^2 K^2 \omega}\int (ns^2 + nc^2)\,du =$$

$$= \frac{1}{A_0^2 \omega}\,(Ns\,u + Nc\,u) \qquad \text{(buckled)} \qquad (3.130)$$

The expressions for Ns, Nd, and Nc are given by Eqs. 3.141.

3.10 The Jacobian Elliptic Functions

Consider the Lagrangian elliptic integral of the first kind (Ref. 7, p. 54)

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2 t^2}} \qquad (3.131)$$

By taking the inverse function one obtains the Jacobian amplitude

$$\varphi = am\, u \qquad (3.132)$$

The Jacobian elliptic functions are defined by(Ref. 7, p. 92)

$$s = sn\,u = \sin\varphi = \sin(am\,u) \qquad (3.133)$$

$$c = cn\,u = \cos\varphi = \cos(am\,u) = \sqrt{1-s^2} \qquad (3.134)$$

$$d = dn\,u = \sqrt{1-k^2\sin^2\varphi} = \sqrt{1-k^2 sn^2 u} \qquad (3.135)$$

These functions are plotted, for convenience, in Figs. 3a, 3b, and 3c. The functions s, c, and d satisfy the following properties (Ref. 7, p. 96)

$$\frac{ds}{du} = cd, \quad \frac{dc}{du} = -sd, \quad \frac{dd}{du} = -k^2 s c \qquad (3.136)$$

It is convenient to introduce the function E (Ref. 7 pp. 97-98)

$$E(u) = \int_0^u dn^2 u\,du = \frac{\tilde{E}}{\tilde{K}} u + zn\,u \tag{3.137}$$

where zn(u) is a periodic function (the Jacobian Zeba function, see Ref. 7, p.98) plotted in Fig. 3d and

$$\begin{aligned} \tilde{K} &= \hat{K}(K) = F(K, \tfrac{\pi}{2}) \\ \tilde{E} &= \tilde{E}(K) = E(K, \tfrac{\pi}{2}) \end{aligned} \tag{3.138}$$

are the complete elliptic integral of first- and second-kind, respectively (Ref. 7, p 54). It is convenient also to complete the elliptic functions by introducing the following functions

$$\begin{aligned} ns &= \frac{1}{sn} & nc &= \frac{1}{cn} & nd &= \frac{1}{dn} \\ sc &= \frac{sn}{cn} & sd &= \frac{sn}{dn} & cd &= \frac{cn}{dn} \\ cs &= \frac{cn}{sn} & ds &= \frac{dn}{sn} & dc &= \frac{dn}{cn} \end{aligned} \tag{3.139}$$

The twelve functions (Glaisdel's functions) defined by Eqs. 3.133, 3.134, 3.135, and 3.139 have very interesting properties, described in Ref. 6. It should be noted that the integral of the square of any of these functions is not expressible in terms of the Jacobian functions. They will be indicated as

$$Pr\,u = \int_0^u pr^2 u\,du \tag{3.140}$$

where pr stands for any combination of s, c, d, and n* (Ref. 6, p. 235). These functions can be expressed in terms of the function E (defined by Eq. 3.136) as follows (Ref. 6, p. 238)

*for example, $Dnu = \int_0^u dn^2\, udu = E(u)$ (see Eq. 3.136).

$$\mathrm{Cs}\,u = -E - \frac{cd}{s}$$

$$\mathrm{Ns}\,u = -E + u + \frac{cd}{s}$$

$$\mathrm{Ds}\,u = -E + K'^2 u - \frac{cd}{s}$$

$$\mathrm{Sc}\,u = \frac{1}{K'^2}\left(-E + \frac{sd}{c}\right)$$

$$\mathrm{Dc}\,u = -E + u + \frac{sd}{c}$$

$$\mathrm{Nc}\,u = \frac{1}{K'^2}\left(-E + K'^2 u + \frac{sd}{c}\right)$$

$$\mathrm{Dn}\,u = E$$

$$\mathrm{Sn}\,u = \frac{1}{K^2}\left(-E + u\right)$$

$$\mathrm{Cn}\,u = \frac{1}{K^2}\left(E - K'^2 u\right)$$

$$\mathrm{Nd}\,u = \frac{1}{K'^2}\left(E - K^2 \frac{sc}{d}\right)$$

$$\mathrm{Cd}\,u = \frac{1}{K^2}\left(-E + u + K^2 \frac{sc}{d}\right)$$

$$\mathrm{Sd}\,u = \frac{1}{K^2 K'^2}\left(E - K'^2 u - K^2 \frac{sc}{d}\right) \tag{3.141}$$

with

$$K' = \sqrt{1 - K^2} \tag{3.142}$$

SECTION 4

SMALL DAMPING TERMS

4.1 Introduction and Summary

In this section, the problem of small damping has been studied. Assume that the damping is of order ε

$$g_n = \varepsilon \bar{g}_n \tag{4.1}$$

Then the governing equation is given by

$$\frac{d^2 W_1}{dt^2} + \varepsilon \bar{g}_1 \frac{dW_1}{dt} + \Omega_1^2 W_1 - \Lambda W_2 + C_{11} W_1^3 + C_{12} W_1 W_2^2 = 0$$

$$\frac{d^2 W_2}{dt^2} + \varepsilon \bar{g}_2 \frac{dW_2}{dt} + \Omega_2^2 W_2 + \Lambda W_1 + C_{21} W_1^2 W_2 + C_{22} W_2^3 = 0 \tag{4.2}$$

Note that with this magnitude of damping, the imaginary part of u in Eq. 2.40 of Ref. 2 is of order ε, hence the real parts of α, β, and γ in Eqs. 2.71, 2.69, and 2.70 of Ref. 2 are also of order ε. Therefore $\partial|A|/\partial t_2$ in Eq. (a) of the footnote following Eq. 2.72 of Ref. 2 is of order ε. This obviously violates the principle of balancing terms of order ε. Therefore, a new scaling is needed. A convenient set of scalings has been chosen in Eqs. 4.3 and 4.4. Note that the "odd" scales, t_1, t_3, . . . , as well as the "even" vectors $\{W_2\}$, $\{W_4\}$ have been introduced. The first-order system, Eq. 4.10, is studied in Subsection 4.3, where it is shown that Λ cannot be determined yet. The secular terms in the second-order system (Subsection 4.4) can be avoided by a suitable choice of λ_F. As a consequence, the solution does not depend upon t_1. Next, in order to avoid secular terms in the third-order system, one obtains the variation with t_2 which is periodic and does not yield any limit cycle. Finally, by avoiding secular terms on the fourth-order system, one obtains an equation in the scale t_3. The real part yields the change of amplitude in time and the limit-cycle type of behavior. But the imaginary part cannot be satisfied unless the steady state has been reached. In other words, secular terms cannot be eliminated completely during transient response. This study shows that although the multiple time scale enables one to see the variation of the amplitude and predict the limit cycle amplitude, it is not general enough to avoid secular terms during transient response. A possible interpretation of the existence of secular terms in the transient response is given in Section 5.

4.2 General Formulation

Having introduced damping terms of order ε, it is convenient to introduce multiple time scales of the form

$$t_n = \varepsilon^n t \qquad n = 0, 1, 2, \cdots \tag{4.3}$$

Similarly

$$\{W\} = \varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \varepsilon^3\{W_3\} + \varepsilon^4\{W_4\} + \cdots \tag{4.4}$$

Note that the "odd" scales t_1, t_3, . . . , and the "even" vectors W_2, W_4, . . . have been included. Combining Eqs. 4.1, 4.3, and 4.4 yields, in matrix form, (see Eqs. 2.2, 2.4, and 2.5)

$$\begin{aligned}
&\left(\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \cdots\right)^2 \left(\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \varepsilon^3\{W_3\} + \varepsilon^4\{W_4\} + \cdots\right) \\
&+ \varepsilon[\bar{G}]\left(\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \cdots\right)\left(\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \varepsilon^3\{W_3\} + \cdots\right) \\
&+ [\Omega^2]\left(\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \varepsilon^3\{W_3\} + \varepsilon^4\{W_4\} + \cdots\right) \\
&+ (\Lambda_0 + \Lambda_2\varepsilon^2)[E]\left(\varepsilon\{W_1\} + \varepsilon^2\{W_2\} + \varepsilon^3\{W_3\} + \cdots\right) \\
&+ \varepsilon^3\{C_3\} + \varepsilon^4\{C_4\} + \cdots = 0
\end{aligned} \tag{4.5}$$

with

$$[G] = \lceil \bar{g}_n \rfloor \tag{4.6}$$

$$\Lambda_2 = \pm 1 \tag{4.7}$$

$$\{C_3\} = \begin{Bmatrix} C_{11} W_{11}^3 + C_{12} W_{11} W_{21}^2 \\ C_{21} W_{11}^2 W_{21}^2 + C_{22} W_{21}^3 \end{Bmatrix} \tag{4.8}$$

$$\{C_4\} = \begin{Bmatrix} C_{11}\, 3 W_{11}^2 W_{12} + C_{12} (W_{12} W_{21}^2 + 2 W_{11} W_{21} W_{22}) \\ C_{21} (W_{22} W_{11}^2 + 2 W_{21} W_{11} W_{12}) + C_{22}\, 3 W_{21}^2 W_{22} \end{Bmatrix} \tag{4.9}$$

Separating terms of the same order yields the following set of systems:

Order ε:

$$\frac{\partial^2 \{W_1\}}{\partial t_0^2} + [\Omega^2]\{W_1\} + \Lambda_0 [E]\{W_1\} = 0 \tag{4.10}$$

Order ε^2:

$$\begin{aligned} &\frac{\partial^2}{\partial t_0^2}\{W_2\} + [\Omega^2]\{W_2\} + \Lambda_0 [E]\{W_2\} \\ &\quad + 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_1\} + [\bar{G}]\frac{\partial}{\partial t_0}\{W_1\} = 0 \end{aligned} \tag{4.11}$$

Order ε^3:

$$\frac{\partial^2}{\partial t_0^2}\{W_3\} + [\Omega^2]\{W_3\} + \Lambda_0 [E]\{W_3\}$$

$$+ 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_2\} + \left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right)\{W_1\}$$

$$+ [\bar{G}]\left(\frac{\partial}{\partial t_0}\{W_2\} + \frac{\partial}{\partial t_1}\{W_1\}\right)$$

$$+ \Lambda_2 [E]\{W_1\} + \{C_3\} = 0$$

(4.12)

Order ε^4:

$$\frac{\partial^2}{\partial t_0^2}\{W_4\} + [\Omega^2]\{W_4\} + \Lambda_0 [E]\{W_4\}$$

$$+ 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_3\} + \left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right)\{W_2\}$$

$$+ \left(2\frac{\partial^2}{\partial t_0 \partial t_3} + 2\frac{\partial^2}{\partial t_1 \partial t_2}\right)\{W_1\}$$

$$+ [\bar{G}]\left(\frac{\partial}{\partial t_0}\{W_3\} + \frac{\partial}{\partial t_1}\{W_2\} + \frac{\partial}{\partial t_2}\{W_1\}\right)$$

$$+ \Lambda_2 [E]\{W_2\} + \{C_4\} = 0$$

(4.13)

4.3 The Vector $\{W_1\}$

Consider, first, the vector $\{W_1\}$ which can be obtained by solving Eq. 4.10 as follows. First, set

$$\mathrm{Det.}\left(p^2[I]+[\Omega^2]+\Lambda_0[E]\right)=0 \tag{4.14}$$

or

$$\begin{vmatrix} p^2+\Omega_1^2 & -\Lambda_0 \\ \Lambda_0 & p^2+\Omega_2^2 \end{vmatrix} = 0 \tag{4.15}$$

This is equivalent to

$$p^4+(\Omega_1^2+\Omega_2^2)p^2+\Omega_1^2\Omega_2^2+\Lambda_0^2=0 \tag{4.16}$$

which yields

$$p^2=-\frac{\Omega_1^2+\Omega_2^2}{2}\pm\sqrt{\left(\frac{\Omega_1^2-\Omega_2^2}{2}\right)^2-\Lambda_0^2} \tag{4.17}$$

Thus, there are four roots. These roots can be written in a more interesting form by introducing the positive parameter $\eta < 1$, such that

$$\Lambda_0^2=(1-\eta^2)\,\frac{(\Omega_1^2-\Omega_2^2)^2}{4} \tag{4.18}$$

Then Eq. 4.17 reduces to

$$\begin{aligned} p^2 &= -\frac{\Omega_1^2+\Omega_2^2}{2}\pm\eta\,\frac{\Omega_2^2-\Omega_1^2}{2} \\ &= -\left(\frac{1\pm\eta}{2}\Omega_1^2+\frac{1\mp\eta}{2}\Omega_2^2\right)=-\omega_\pm^2 \end{aligned} \tag{4.19}$$

with

$$\omega_+^2 = \frac{1+\eta}{2}\,\Omega_1^2 + \frac{1-\eta}{2}\,\Omega_2^2$$

$$\omega_-^2 = \frac{1-\eta}{2}\,\Omega_1^2 + \frac{1+\eta}{2}\,\Omega_2^2 \tag{4.20}$$

Thus, the solution for Eq. 4.10 is given by

$$\{W_1\} = 2\,\mathrm{Real}\left[A_+\{U_+\}\,e^{i\omega_+ t} + A_-\{U_-\}\,e^{i\omega_- t}\right] \tag{4.21}$$

where A_+ and A_- are functions of t_1, t_2, and so on, and furthermore

$$\{U_\pm\} = \begin{Bmatrix} 1 \\ u_\pm \end{Bmatrix} \tag{4.22}$$

is the eigenvector of the equation

$$\left(-\omega_\pm^2\,[I] + [\Omega^2] + \Lambda_0\,[E]\right)\{U_\pm\} = 0 \tag{4.23}$$

or, in explicit form (see Eq. 4.2)

$$(\Omega_1^2 - \omega_\pm^2) - \Lambda_0\, U_\pm = 0$$

$$\Lambda_0 + (\Omega_2^2 - \omega_\pm^2)\, U_\pm = 0 \tag{4.24}$$

or, using only the first equation

$$U_\pm = \frac{\Omega_1^2 \pm \omega_\pm^2}{\Lambda_0} = \left[\Omega_1^2\left(1 - \frac{1\pm\eta}{2}\right) - \left(\frac{1\mp\eta}{2}\right)\Omega_2^2\right]$$

$$\times\left[(1-\eta^2)\left(\frac{\Omega_2^2-\Omega_1^2}{2}\right)^2\right]^{-1/2} = -\frac{1\mp\eta}{\sqrt{1-\eta^2}} = -\sqrt{\frac{1\mp\eta}{1\pm\eta}} \tag{4.25}$$

or

$$u_+ = -\sqrt{\frac{1-\eta}{1+\eta}} = u \tag{4.26}$$

$$u_- = -\sqrt{\frac{1+\eta}{1-\eta}} = \frac{1}{u} \tag{4.27}$$

Note that the value of η cannot be determined from the system of order ε. As is shown in the next section, the value of η (and thus of Λ_o) is determined by the effect of damping terms on the system of order ε^2.

4.4 The Vector $\{W_2\}$ and the Relation $\partial A/\partial t_1 = 0$

Consider the vector $\{W_2\}$, which can be obtained by solving Eqs. 4.11 and 4.21, yielding

$$\frac{\partial^2}{\partial t_0^2}\{W_2\} + [\Omega^2]\{W_2\} + \Lambda_o[E]\{W_2\} + \{Z_2\} = 0 \tag{4.28}$$

with

$$\begin{aligned}\{Z_2\} &= 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_1\} + [\bar{G}]\frac{\partial}{\partial t_0}\{W_1\} \\ &= 2\,\mathrm{Real}\Big[i\omega_+\Big(2\{U_+\}\frac{\partial A_+}{\partial t_1} + [\bar{G}]\{U_+\}A_+\Big)e^{i\omega_+ t_0} \\ &\quad + i\omega_-\Big(2\{U_-\}\frac{\partial A_-}{\partial t_1} + [\bar{G}]\{U_-\}A_-\Big)e^{i\omega_- t_0}\Big]\end{aligned} \tag{4.29}$$

In order to avoid secular terms, the conditions*

$$2\lfloor U_\pm^L \rfloor\{U_\pm\}\frac{\partial A_\pm}{\partial t_1} + \lfloor U_\pm^L \rfloor[\bar{G}]\{U_\pm\}A_\pm = 0 \tag{4.30}$$

*Where, as usual, $\lfloor U_\pm^L \rfloor = \lfloor 1, -u_\pm \rfloor$

must be satisfied. This yields

$$A_{\pm} = A_{o\pm}\, e^{-\sigma_{\pm} t_1} \tag{4.31}$$

with

$$\sigma_{\pm} = \frac{\lfloor U_{\pm}^{b} \rfloor [\bar{G}] \{U_{\pm}\}}{2 \lfloor U_{\pm}^{b} \rfloor \{U_{\pm}\}} = \frac{1}{2} \frac{\bar{g}_1 - \bar{g}_2 u_{\pm}^2}{1 - u_{\pm}^2} \tag{4.32}$$

Assume that

$$\bar{g}_2 > \bar{g}_1 \tag{4.33}$$

Then the use of Eq. 4.27 yields

$$\sigma_{-} > 0 \tag{4.34}$$

whereas the use of Eq. 4.26 yields

$$\sigma_{+} \gtreqless 0 \quad \text{for} \quad u_{+}^2 \lesseqgtr \bar{g}_1/\bar{g}_2 \tag{4.35}$$

Thus the solution becomes unstable for $u_+^2 > \bar{g}_1/g_2$. Hence, it is convenient to choose η such that the solution is at the limit of the stability:

$$u_{+}^2 = \frac{\bar{g}_1}{\bar{g}_2} \qquad (\sigma_{+} = 0) \tag{4.36}$$

or (see Eq. 4.27)

$$\eta = \frac{\bar{g}_2 - \bar{g}_1}{\bar{g}_2 + \bar{g}_1} \tag{4.37}$$

which implies (see Eq. 4.18)

$$\Lambda_o = \frac{\bar{g}_1 \bar{g}_2}{(\bar{g}_1 + \bar{g}_2)^2} (\Omega_1^2 - \Omega_2^2)^2 \tag{4.38}$$

In the following, the solution at the value of Λ_o given in Eq. 4.38 is considered. Furthermore, for the sake of simplicity, the term of Eq. 4.35 which contains A_- is dropped since A_- is exponentially damped. Thus, Eq. 4.21 can be rewritten as

$$\{W_1\} = A\{u\} e^{i\omega t_0} + A^*\{u^*\} e^{-i\omega t_0} \tag{4.39}$$

where A is a function of t_2, t_3, and so on. Therefore

$$\frac{\partial A}{\partial t_1} = 0 \tag{4.40}$$

Furthermore

$$\omega^2 = \omega_F^2 = \frac{\bar{g}_2}{\bar{g}_1 + \bar{g}_2} \Omega_1^2 + \frac{\bar{g}_1}{\bar{g}_1 + \bar{g}_2} \Omega_2^2 \tag{4.41}$$

and

$$\{u\} = \begin{Bmatrix} 1 \\ \bar{u} \end{Bmatrix} \tag{4.42}$$

with

$$\bar{u} = -\sqrt{\frac{\bar{g}_1}{\bar{g}_2}} \tag{4.43}$$

Note that by using Eq. 4.1 and neglecting higher order terms, these results agree exactly with those obtained in Ref. 2, Section 2.

Finally, Eq. 4.28 can be solved to yield

$$\{W_2\} = 2\,\mathrm{Real}\left[\left(B\{u\} + A\{V\}\right) e^{i\omega t_0}\right] \tag{4.44}$$

with B a function of t_1, t_2 and so on, and

$$\{V\} = i\omega[N][G]\{u\} = \begin{Bmatrix} 0 \\ v \end{Bmatrix} \tag{4.45}$$

with

$$[N] = -\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{-\omega^2 + \Omega_2^2} \end{bmatrix} \tag{4.46}$$

and (see Eq. 4.41)

$$\bar{v} = i\omega \frac{-1}{-\omega^2 + \Omega_2^2} \bar{g}_2 \bar{u} = -i \frac{\bar{g}_2 \omega}{\Omega_2^2 - \omega^2} \bar{u}$$

$$= -i \frac{(\bar{g}_1 + \bar{g}_2)\bar{u}\,\omega_F}{\Omega_2^2 - \Omega_1^2} \tag{4.47}$$

Note that terms with $e^{i\omega_- t_o}$ have been dropped since it can be shown that they are also exponentially damped.

Finally, the condition $\sigma_+ = 0$, (see Eq. 4.36) which defines the value of λ_o, can be rewritten as (see Eq. 4.42)

$$\lfloor U^L \rfloor [\bar{G}] \{U\} = 0 \tag{4.48}$$

4.5 The Vector $\{W_3\}$ and the Function $A(t_2)$

Consider the vector $\{W_3\}$. Combining Eqs. 4.12, 4.39, and 4.43 yields

$$\frac{\partial^2}{\partial t_0^2}\{W_3\} + [\Omega^2]\{W_3\} + \Lambda_0 [E]\{W_3\} + \{Z_3\} = 0 \tag{4.49}$$

with (noting that $\partial A/\partial t_1 = 0$, see Eq. 4.40)

$$\begin{aligned}
\{Z_3\} &= 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_2\} + \left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right)\{W_1\} \\
&\quad + [\bar{G}]\left(\frac{\partial}{\partial t_0}\{W_2\} + \frac{\partial}{\partial t_1}\{W_1\}\right) + \Lambda_2 [E]\{W_1\} \\
&\quad + \{C_3\} \\
&= 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_2\} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\{W_1\} + [\bar{G}]\frac{\partial}{\partial t_0}\{W_2\} \\
&\quad + \Lambda_2 [E]\{W_1\} + \{C_3\} \\
&= 2\,\mathrm{Real}\left[\{Z_3^{(1)}\} e^{i\omega t_0} + \{Z_3^{(3)}\} e^{i3\omega t_0}\right]
\end{aligned} \tag{4.50}$$

where by using the explicit expression for $\{C_3\}$ (see Eq. 4.112):

$$\{Z_3^{(1)}\} = 2i\omega \frac{\partial B}{\partial t_1}\{U\} + 2i\omega \frac{\partial A}{\partial t_2}\{U\}$$

$$+ i\omega[\bar{G}](B\{U\} + A\{V\})$$

$$+ \Lambda_2[E]A\{U\} + \{H_1\}A^2A^* \tag{4.51}$$

$$\{Z_3^{(3)}\} = \{H_0\}A^3 \tag{4.52}$$

In order to avoid secular terms the condition

$$\lfloor U^L \rfloor \{Z_3^{(1)}\} = 0 \tag{4.53}$$

must be satisfied. By making use of Eq. 4.48, this yields

$$2i\omega\lfloor U^L\rfloor\{U\}\left(\frac{\partial B}{\partial t_1} + \frac{\partial A}{\partial t_2}\right) + i\omega\lfloor U^L\rfloor[G]\{V\}A$$

$$+\Lambda_2\lfloor U^L\rfloor[E]\{U\}A + \lfloor U^L\rfloor\{H_1\}A^2A^* = 0 \tag{4.54}$$

This equation yields

$$\frac{\partial B}{\partial t_1} = 0 \tag{4.55}$$

and

$$\frac{\partial A}{\partial t_2} + \bar{\beta}A + \bar{\gamma}A^2A^* = 0 \tag{4.56}$$

where

$$\bar{\beta} = \frac{1}{\alpha}\left(i\omega\lfloor U^L\rfloor[G]\{V\} + \Lambda_2\lfloor U^L\rfloor[E]\{U\}\right) \tag{4.57}$$

$$\bar{\gamma} = \frac{1}{\bar{\alpha}} \lfloor U^L \rfloor \{H_1\} \tag{4.58}$$

with

$$\bar{\alpha} = 2i\omega \lfloor U^L \rfloor \{U\} \tag{4.59}$$

In explicit form

$$\bar{\alpha} = i2\omega(1-\bar{u}^2)$$

$$\bar{\beta} = \frac{1}{\bar{\alpha}}\left[i\omega\,(-\bar{u}\,\bar{g}_2\bar{v}) + \Lambda_2(-2\bar{u})\right]$$

$$\bar{\gamma} = \frac{1}{\bar{\alpha}}\,3\left[C_{11} + (C_{12} - C_{21})\bar{u}^2 - C_{22}\bar{u}^4\right] \tag{4.60}$$

From Eq. 4.60, it is obvious that $\bar{\alpha}$, $\bar{\beta}$,and $\bar{\gamma}$ are imaginary. That is

$$Real\ \bar{\alpha} = 0$$

$$Real\ \bar{\beta} = 0$$

$$Real\ \bar{\gamma} = 0 \tag{4.61}$$

Thus by setting

$$A = |A|\, e^{i\varphi} \tag{4.62}$$

Equation 4.56 yields

$$\frac{\partial |A|}{\partial t_2} = 0 \tag{4.63}$$

and

$$\frac{\partial \varphi}{\partial t_2} + \bar{\beta}_I + \bar{\gamma}_I\, |A|^2 = 0 \tag{4.64}$$

where the following relations have been used (see Eq. 4.61)

$$\begin{aligned} \bar{\beta} &= i\bar{\beta}_I \\ \bar{\gamma} &= i\bar{\gamma}_I \end{aligned} \tag{4.65}$$

Thus A is given by Eq. 4.62 with $|A|$ independent of t_2 and

$$\varphi = -(\bar{\beta}_I + \bar{\gamma}_I |A|^2)\, t_2 + \varphi_0 \tag{4.66}$$

with ϕ_o a function of t_3, t_4, and so on.

Finally, since Eqs. 4.55 and 4.56 are satisfied, then Eq. 4.53 is also satisfied and the vector $\{W_3\}$ does not contain secular terms and is given by

$$\{W_3\} = \left(C\{u\} + \{P_3^{(1)}\}\right) e^{i\omega t_0} + \{P_3^{(3)}\}\, e^{i3\omega t_0} \tag{4.67}$$

with

$$\{P_3^{(1)}\} = [N]\{Z_3^{(1)}\} \tag{4.68}$$

$$\{P_3^{(3)}\} = -\left[-9\omega^2[I] + [\Omega^2] + \Lambda_2[E]\right]^{-1} \{Z_3^{(3)}\} \tag{4.69}$$

Note that, according to Eqs. 4.40 and 4.55

$$\frac{\partial}{\partial t_1}\{W_3\} = \frac{\partial C}{\partial t_1}\{u\}\, e^{i\omega t_0} + \frac{\partial C^*}{\partial t_1}\{u^*\}\, e^{-i\omega t_0} \tag{4.70}$$

and that combining Eqs. 4.51, 4.55, and 4.68

$$\{P_3^{(1)}\} = 2i\omega\,[N]\{u\}\frac{\partial A}{\partial t_2} + [N]\left(i\omega\,[\bar{G}]\{V\} + \Lambda_2[E]\{u\}\right)A$$
$$+ [N]\{H_1\}A^2A^* + i\omega\,[N][\bar{G}]\{u\}B$$
$$= \{P_\alpha\}\frac{\partial A}{\partial t_2} + \{P_\beta\}A + \{P_\gamma\}A^2A^* + \{V\}B \tag{4.71}$$

with {V} given by Eq. 4.45 and

$$\{P_\alpha\} = 2i\omega\,[N]\{u\}$$
$$\{P_\beta\} = [N]\left(i\omega[\bar{G}]\{V\} + \Lambda_2[E]\{u\}\right)$$
$$= [N]\left(-\omega^2[\bar{G}][N][\bar{G}] + \Lambda_2[E]\right)\{u\}$$
$$\{P_\gamma\} = [N]\{H_1\} \tag{4.72}$$

Note the similarity of the definition of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ (Eqs. 4.57 to 4.59) and $\{P_\alpha\}$, $\{P_\beta\}$ and $\{P_\gamma\}$.

4.6 The Vector $\{W_4\}$ and the Equation for B

Consider the vector $\{W_4\}$. Combining Eqs. 4.13, 4.39, 4.44, and 4.67 yields, by taking into account that $\partial A/\partial t_1 = 0$ (Eq. 4.40 and $\partial B/\partial t_1 = 0$ (Eq. 4.55 and Eq. 4.70),

$$\frac{\partial^2}{\partial t_0^2}\{W_4\} + [\Omega^2]\{W_4\} + \Lambda_0[E]\{W_4\} + \{Z_4\} = 0 \tag{4.73}$$

with ($\{C_4\}$ is explicitly defined in Eq. 4.118)

$$\{Z_4\} = 2\frac{\partial^2}{\partial t_0 \partial t_1}\{W_3\} + \left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right)\{W_2\}$$

$$+ \left(2\frac{\partial^2}{\partial t_0 \partial t_3} + 2\frac{\partial^2}{\partial t_1 \partial t_2}\right)\{W_1\} + [\bar{G}]\left(\frac{\partial}{\partial t_0}\{W_3\}\right.$$

$$\left. + \frac{\partial}{\partial t_1}\{W_2\} + \frac{\partial}{\partial t_2}\{W_1\}\right) + \Lambda_2[E]\{W_2\} + \{C_4\}$$

$$= 2\,\mathrm{Real}\Big[2i\omega \frac{\partial C}{\partial t_1}\{U\}e^{i\omega t_0} + 2i\omega\left(\frac{\partial B}{\partial t_2}\{U\} + \frac{\partial A}{\partial t_2}\{V\}\right)e^{i\omega t_0}$$

$$+ 2i\omega \frac{\partial A}{\partial t_3}\{U\}e^{i\omega t_0} + i\omega[\bar{G}](C\{U\} + P_3^{(1)})e^{i\omega t_0}$$

$$+ i3\omega[\bar{G}][P_3^{(3)}]e^{i3\omega t_0} + [G]\{U\}\frac{\partial A}{\partial t_2}e^{i\omega t_0}$$

$$+ \Lambda_2[E](\{U\}B + \{V\}A)e^{i\omega t_0} + (\{H_1\}A^2B$$

$$+ \{K\}A^3)e^{i3\omega t_0} + (\{H_1\}(2AA^*B + A^2B^*)$$

$$+ \{K\}A^2A^*)e^{i\omega t_0}\Big] \tag{4.74}$$

or

$$\{Z_4\} = Z_4^{(1)}e^{i\omega t_0} + Z_4^{(3)}e^{i3\omega t_0} + Z_4^{(1)*}e^{-i\omega t_0} + Z_4^{(3)*}e^{-i3\omega t_0} \tag{4.75}$$

with

$$Z_4^{(3)} = i3\omega[\bar{G}]\{P_3^{(3)}\} + \{H_1\}A^2B + \{K\}\{A^3\} \tag{4.76}$$

and (see Eq. 4.69)

$$
\begin{aligned}
Z_4^{(1)} &= 2i\omega\left(\frac{\partial C}{\partial t_1}+\frac{\partial B}{\partial t_2}+\frac{\partial A}{\partial t_3}\right)\{U\}+2i\omega\frac{\partial A}{\partial t_2}\{V\} \\
&\quad + i\omega[\bar{G}]\Big[C\{U\}+\{P_\alpha\}\frac{\partial A}{\partial t_2}+\{P_\beta\}A \\
&\quad + \{P_\gamma\}A^2A^* + \{V\}B\Big]+[\bar{G}]\{U\}\frac{\partial A}{\partial t_2}+\Lambda_2[E] \\
&\quad \times\Big(\{U\}B+\{V\}A\Big)+\{H_1\}(2AA^*B+A^2B^*)+\{K\}A^2A^* \\
&= 2i\omega\left(\frac{\partial C}{\partial t_1}+\frac{\partial B}{\partial t_2}+\frac{\partial A}{\partial t_3}\right)\{U\}+\Big(i\omega[\bar{G}]\{V\} \\
&\quad +\Lambda_2[E]\{U\}\Big)B+\{H_1\}(2AA^*B+A^2B^*) \\
&\quad +\Big(2i\omega\{V\}+i\omega[\bar{G}]\{P_\alpha\}+[\bar{G}]\{U\}\Big)\frac{\partial A}{\partial t_2} \\
&\quad +\Big(i\omega[\bar{G}]\{P_\beta\}+\Lambda_2[E]\{V\}\Big)A \\
&\quad +\Big(i\omega[\bar{G}]\{P_\gamma\}+\{K\}\Big)A^2A^* \\
&\quad + i\omega[\bar{G}]\{U\}C
\end{aligned}
\tag{4.77}
$$

In order to avoid secular terms for $\{w_4\}$, the condition

$$\lfloor U^L \rfloor \{Z_4^{(1)}\} = 0 \tag{4.78}$$

must be satisfied. Combining Eqs. 4.48, 4.77, and 4.78 yields

$$\bar{\alpha}\left[\left(\frac{\partial C}{\partial t_1} + \frac{\partial B}{\partial t_2} + \frac{\partial A}{\partial t_3}\right) + \bar{\beta} B + \bar{\gamma}\left(2AA^*B + A^2B^*\right)\right]$$

$$+ \bar{\alpha}'\left[\frac{\partial A}{\partial t_2} + \bar{\beta}' A + \bar{\gamma}' A^2 A^*\right] = 0 \tag{4.79}$$

with $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$ given by Eqs. 4.57 to 4.59 and

$$\bar{\alpha}' = \lfloor U^L \rfloor \left(2i\omega\{V\} + i\omega[\bar{G}]\{P_\alpha\}\right)$$

$$= -2\omega^2 \lfloor U^L \rfloor \left([N][\bar{G}] + [\bar{G}][N]\right)\{U\} \tag{4.80}$$

$$\bar{\beta}' = \frac{1}{\bar{\alpha}'} \lfloor U^L \rfloor \left(\Lambda_2[E]\{V\} + i\omega[\bar{G}]\{P_\beta\}\right)$$

$$= \frac{1}{\bar{\alpha}'} \lfloor U^L \rfloor \left[i\omega\Lambda_2\left([E][N][\bar{G}] + [\bar{G}][N][E]\right) - i\omega^3[\bar{G}][N][\bar{G}][N][\bar{G}]\right] \tag{4.81}$$

$$\bar{\gamma}' = \frac{1}{\bar{\alpha}'} \lfloor U^L \rfloor \left(\{K\} + i\omega[\bar{G}]\{P_\gamma\}\right)$$

$$= \frac{1}{\bar{\alpha}'} \lfloor U^L \rfloor \left(\{K\} + i\omega[\bar{G}][N]\{H_1\}\right) \tag{4.82}$$

Equation 4.80 is satisfied by

$$\frac{\partial C}{\partial t_1} = 0 \tag{4.83}$$

and

$$\bar{\alpha}\left[\left(\frac{\partial B}{\partial t_2}+\frac{\partial A}{\partial t_3}\right)+\bar{\beta}B+\bar{\gamma}\left(2AA^*B\right.\right.$$
$$\left.\left.+A^2B^*\right)\right]+\bar{\alpha}'\left[\frac{\partial A}{\partial t_2}+\bar{\beta}'A\right.$$
$$\left.+\bar{\gamma}'A^2A^*\right]=0$$

(4.84)

The solution for Eq. 4.84 is discussed in Subsection 4.7. Once this equation is satisfied, the solution for Eq. 4.73 is given by

$$\{W_4\}=\left(D\{u\}+\{P_4^{(1)}\}\right)e^{i\omega t_0}+\{P_4^{(3)}\}e^{i3\omega t_0} \tag{4.85}$$

with

$$\{P_4^{(1)}\}=[N]\{Z_4^{(1)}\} \tag{4.86}$$

and

$$\{P_4^{(3)}\}=-\left[-9\omega^2[I]+[\Omega^2]+\Lambda_0[E]\right]^{-1}\{Z_4^{(3)}\} \tag{4.87}$$

4.7 The Functions $B(t_2)$ and $A(t_2,t_3)$

In this subsection, the solution for Eq. 4.84 is discussed. As shown below, this yields the dependence of B upon t_2 and of A upon t_3. Letting

$$B=bA \tag{4.88}$$

and combining with Eq. 4.84 yields

$$\bar{\alpha}\left[\left(b\frac{\partial A}{\partial t_2}+A\frac{\partial b}{\partial t_2}\right)+\bar{\beta}Ab+\bar{\gamma}(2A^2A^*b+A^2A^*b^*)\right]$$

$$+\bar{\alpha}\frac{\partial A}{\partial t_3}+\bar{\alpha}'\left[\frac{\partial A}{\partial t_2}+\bar{\beta}'A+\bar{\gamma}'A^2A^*\right]=0 \tag{4.89}$$

or by using the differential equation for A (Eq. 4.56)

$$\bar{\alpha}\left[A\frac{\partial b}{\partial t_2}+\bar{\gamma}A^2A^*(b+b^*)\right]+\bar{\alpha}\frac{\partial A}{\partial t_3}+\bar{\alpha}'(\bar{\beta}'-\bar{\beta})A$$

$$+\bar{\alpha}'(\bar{\gamma}'-\bar{\gamma})A^2A^*=0 \tag{4.90}$$

Note that $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$ are imaginary (Eq. 4.61). On the other hand, according to Eqs. 4.80 to 4.82, $\bar{\alpha}'$ is real; whereas

$$\bar{\beta}'=i\bar{\beta}_I'$$

$$\bar{\gamma}'=i\bar{\gamma}_I' \tag{4.91}$$

are imaginary. Thus, Eq. 4.90 can be rewritten as

$$\frac{\partial b}{\partial t_2}+\bar{\gamma}AA^*(b+b^*)+\frac{\partial}{\partial t_3}\ln A+\tilde{\beta}+\tilde{\gamma}AA^*=0 \tag{4.92}$$

with

$$\tilde{\beta}=\frac{\bar{\alpha}'}{\bar{\alpha}}(\bar{\beta}'-\bar{\beta})=\frac{\bar{\alpha}'}{\bar{\alpha}_I}(\bar{\beta}_I'-\bar{\beta}_I)$$

$$\tilde{\gamma}=\frac{\bar{\alpha}'}{\bar{\alpha}}(\bar{\gamma}'-\bar{\gamma})=\frac{\bar{\alpha}'}{\bar{\alpha}_I}(\bar{\gamma}_I'-\bar{\gamma}_I) \tag{4.93}$$

Note that*

$$\frac{\partial}{\partial t_3}\ln A=\frac{\partial}{\partial t_3}\ln|A|+i\frac{\partial\varphi}{\partial t_3} \tag{4.94}$$

*According to Eq. 4.62, $\ln A = \ln|A| + i(\phi + 2n\pi)$.

Thus, by setting

$$b = b_R + i b_I \tag{4.95}$$

and by separating real and imaginary parts, one obtains

$$\frac{\partial b_R}{\partial t_2} + \frac{\partial}{\partial t_3} \ln |A| + \tilde{\beta} + \tilde{\gamma} |A|^2 = 0 \tag{4.96}$$

$$\frac{\partial b_I}{\partial t_2} + 2 \tilde{\gamma}_I |A|^2 b_R + \frac{\partial \varphi}{\partial t_3} = 0 \tag{4.97}$$

In order to avoid secular terms for b_R, the condition

$$\frac{\partial}{\partial t_3} \ln |A| + \tilde{\beta} + \tilde{\gamma} |A|^2 = 0 \tag{4.98}$$

must be satisfied. Then Eq. 4.96 reduces to

$$\frac{\partial b_R}{\partial t_2} = 0 \tag{4.99}$$

which implies that b_R is an arbitrary function of t_3, t_4, and so on. On the other hand, Eq. 4.98 can be rewritten as

$$\frac{\partial |A|}{\partial t_3} + \tilde{\beta} |A| + \tilde{\gamma} |A|^3 = 0 \tag{4.100}$$

The solution for Eq. 4.100 is given by

$$|A| = \left(- \frac{\tilde{\beta}}{\tilde{\gamma}} + K e^{2\tilde{\beta} t_3} \right)^{-1/2} \tag{4.101}$$

where k is an arbitrary function of t_4, t_5, . . .

Next, consider Eq. 4.97, which can be rewritten as (see Eq. 4.66)

$$\frac{\partial b_I}{\partial t_2} + 2\bar{\gamma}_I |A|^2 b_R - 2\bar{\gamma}_I t_2 |A| \frac{\partial |A|}{\partial t_3} + \frac{\partial \varphi_0}{\partial t_3} = 0 \tag{4.102}$$

This equation yields secular terms (of type t_2) unless

$$\frac{\partial \varphi_0}{\partial t_3} = -2\bar{\gamma}_I |A|^2 b_R \tag{4.103}$$

and "super-secular terms" (of type t_2^2) unless

$$2\bar{\gamma}_I |A| \frac{\partial |A|}{\partial t_3} = 0 \tag{4.104}$$

Equation 4.103 is satisfied by

$$b_R \equiv 0 \tag{4.105}$$

and

$$\frac{\partial \varphi_0}{\partial t_3} = 0 \tag{4.106}$$

On the other hand, Eq. 4.104 can be satisfied only if

$$\frac{\partial |A|}{\partial t_3} = 0 \tag{4.107}$$

(steady-state case) since $\bar{\gamma}_I \neq 0$. Then Eq. 4.102 reduces to

$$\frac{\partial b_I}{\partial t_2} = 0 \tag{4.108}$$

which implies that b_I is an arbitrary function of t_3, t_4, and so on. In particular, one can choose

$$b_I = 0 \tag{4.109}$$

Summarizing, Eq. 4.74 can be solved by

$$b \equiv 0 \tag{4.110}$$

if the condition

$$A = \sqrt{-\tilde{\beta}/\tilde{\gamma}} \qquad \text{(limit cycle)} \tag{4.111}$$

is satisfied. On the other hand, during the transient response the multiple time scaling technique can predict the variation of the amplitude $|A|$ but is not general enough to eliminate the secular terms by a suitable change of the frequency with time. An interpretation of this result is given in Subsection 4.9.

4.8 The Nonlinear Terms

In this subsection an explicit expression for $\{C_3\}$ and $\{C_4\}$ is derived. Consider first, $\{C_3\}$. Combining Eqs. 4.8 and 4.39 yields

$$\begin{aligned}\{C_3\} &= \begin{Bmatrix} C_{11} W_{11}^3 + C_{12} W_{11} W_{12}^2 \\ C_{21} W_{11}^2 W_{21} + C_{22} W_{21}^3 \end{Bmatrix} \\ &= A^3 e^{i3\omega t_0} \begin{Bmatrix} C_{11} + C_{12} U^2 \\ C_{21} U + C_{22} U^3 \end{Bmatrix} + 3A^2 A^* e^{i\omega t_0} \begin{Bmatrix} C_{11} + C_{12} U^2 \\ C_{21} U + C_{22} U^3 \end{Bmatrix} \\ &= A^3 e^{i3\omega t_0} \{H_0\} + A^2 A^* e^{i\omega t_0} \{H_1\} \end{aligned} \tag{4.112}$$

with

$$\{H_0\} = \begin{Bmatrix} C_{11} + C_{12} U^2 \\ C_{21} U + C_{22} U^3 \end{Bmatrix} \tag{4.113}$$

$$\{H_1\} = 3\{H_0\} \tag{4.114}$$

Next, consider $\{C_4\}$; combining Eqs. 4.9, 4.39, and 4.44 yields

$$\begin{aligned}\{C_4\} &= \begin{Bmatrix} C_{11} 3 W_{11}^2 W_{12} + C_{12} (W_{12} W_{21}^2 + 2 W_{11} W_{21} W_{22}) \\ C_{21} (2 W_{11} W_{12} W_{21} + W_{11}^2 W_{22}) + C_{22} 3 W_{21}^2 W_{22} \end{Bmatrix} \\ &= \begin{Bmatrix} \zeta_1 \\ \zeta_2 \end{Bmatrix} + \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} \end{aligned} \tag{4.115}$$

where, setting for convenience

$$\omega t_\circ = \tau \tag{4.116}$$

ξ_1, ξ_2, ζ_1 and ζ_2 are given by (note that $u^* = u$ and $v^* = -v$)

$$\begin{aligned}
\xi_1 &= C_{11}\, 3\,(Ae^{i\tau}+A^*e^{-i\tau})^2(Be^{i\tau}+B^*e^{-i\tau}) \\
&\quad + C_{12}\big[(Be^{i\tau}+B^*e^{-i\tau})(Au\,e^{i\tau}+A^*u\,e^{-i\tau})^2 \\
&\quad + 2(Ae^{i\tau}+A^*e^{-i\tau})(Aue^{i\tau}+A^*u\,e^{-i\tau})(Bue^{i\tau}+B^*u\,e^{-i\tau})\big] \\
&= (C_{11}+C_{12}u^2)\,3\,(Ae^{i\tau}+A^*e^{-i\tau})^2(Be^{i\tau}+B^*e^{-i\tau}) \\
&= 2Re.\big[3\,(C_{11}+C_{12}u^2)\,[A^2B\,e^{i3\tau}+(2AA^*B+A^2B^*)e^{i\tau}]\big]
\end{aligned}$$

$$\begin{aligned}
\xi_2 &= C_{21}\,[2\,(Ae^{i\tau}+A^*e^{-i\tau})(Be^{i\tau}+B^*e^{-i\tau})(Aue^{i\tau}+A^*u\,e^{-i\tau}) \\
&\quad + (Ae^{i\tau}+A^*e^{-i\tau})^2(Bue^{i\tau}+B^*u\,e^{-i\tau})] \\
&\quad + C_{22}\,3\,(Au\,e^{i\tau}+A^*u\,e^{-i\tau})^2(Bue^{i\tau}+B^*ue^{-i\tau}) \\
&= (C_{21}u+C_{22}u^3)\,3\,(Ae^{i\tau}+A^*e^{i\tau})^2(Be^{i\tau}+B^*e^{-i\tau}) \\
&= 2Re.\big[3\,(C_{21}u+C_{22}u^3)\,[A^2B\,e^{i\tau}+(2AA^*B+A^2B^*)e^{i\tau}]\big]
\end{aligned}$$

$$\begin{aligned}
\zeta_1 &= 2C_{12}(Ae^{i\tau}+A^*e^{-i\tau})(Aue^{i\tau}+A^*u\,e^{-i\tau})(Ave^{i\tau}-A^*v\,e^{-i\tau}) \\
&= 2C_{12}\,uv\,(Ae^{i\tau}+A^*e^{-i\tau})^2(Ae^{i\tau}-A^*e^{-i\tau}) \\
&= 2Re.\big[2\,C_{12}\,uv\,(A^3e^{i3\tau}+A^2A^*e^{i\tau})\big]
\end{aligned}$$

$$\begin{aligned}
\zeta_2 &= C_{21}\,(Ae^{i\tau}+A^*e^{-i\tau})^2(Ave^{i\tau}-A^*v\,e^{-i\tau}) \\
&\quad + 3\,C_{22}(Au\,e^{i\tau}+A^*ue^{-i\tau})^2(Av\,e^{i\tau}-A^*ve^{-i\tau}) \\
&= 2Re.\big[(C_{12}+3\,C_{22}u^2)\,v\,(A^3e^{i3\tau}+A^2A^*e^{i\tau})\big]
\end{aligned} \tag{4.117}$$

Collecting terms

$$\{C_4\} = 2\,\mathrm{Real}\Big[\{H_1\}\left[A^2 B\, e^{i3\omega t_o} + (2AA^*B + A^2B^*)\, e^{i\omega t_o}\right] + \{K\}\left(A^3 e^{i3\omega t_o} + A^2A^*\, e^{i\omega t_o}\right)\Big] \tag{4.118}$$

with $\{H_1\}$ given by Eq. 4.114 and

$$\{K\} = \begin{Bmatrix} C_{12}\, 2u \\ C_{21} + 3C_{22}u^2 \end{Bmatrix} v = \frac{\partial}{\partial u}\{H_o\}\, v \tag{4.119}$$

4.9 Comparison with Existing Results

Consider the amplitude of the limit cycle given by Eq. 4.111. Combining Eq. 4.111 and Eq. 4.93 yields

$$A^2 = -\frac{\beta' - \beta}{\gamma' - \gamma} \tag{4.120}$$

where β, γ, β' and γ' are given by Eqs. 4.81 and 4.82. Note that Eqs. 4.80 to 4.82 can be rewritten in simple form as

$$\begin{aligned} \bar{\alpha}' &= 2i\omega\left(\lfloor V^L\rfloor\{u\} + \lfloor u^L\rfloor\{V\}\right) \\ \bar{\beta}' &= \frac{1}{\bar{\alpha}'}\left[\Lambda_2\left(\lfloor V^L\rfloor[E]\{u\} + \lfloor u^L\rfloor[E]\{V\}\right) + i\omega\lfloor V^L\rfloor[G]\{V\}\right] \\ \bar{\gamma}' &= \frac{1}{\bar{\alpha}'}\left(\lfloor u^L\rfloor\{K\} + \lfloor V^L\rfloor\{H_1\}\right) \end{aligned} \tag{4.121}$$

where

$$\lfloor V^L\rfloor = i\omega\lfloor u^L\rfloor[\bar{G}][N] = \lfloor 0, -v\rfloor \tag{4.122}$$

with v given by Eq. 4.47.

In explicit form

$$\bar{\alpha}' = -4 i \omega \bar{u} \bar{v} = \frac{\partial \alpha}{\partial u} \bar{v}$$

$$\bar{\alpha}' \bar{\beta}' = -\Lambda_2 2 \bar{v} + i \omega \bar{g}_2 \bar{v}^2 = \frac{\partial \beta}{\partial u} \bar{v}$$

$$\bar{\alpha}' \bar{\gamma}' = 2 \bar{u} \bar{v} (C_{12} - 2 C_{21} - 3 C_{22} \bar{u}^2) \tag{4.123}$$

Combining Eqs. 4.60 and 4.123 yields

$$\alpha \alpha' (\beta' - \beta) = 2 i \omega \bar{v} (-2 \Lambda_2 - i \omega \bar{g}_2 \bar{v}) (1 + \bar{u}^2) \tag{4.124}$$

$$\alpha \alpha' (\gamma' - \gamma) = 4 i \omega \bar{u} \bar{v} \{ 3 C_{11} + (1 + 2 \bar{u}^2) C_{12} - (2 + \bar{u}^2) C_{21} - 3 C_{22} \bar{u}^2 \} \tag{4.125}$$

Thus combining Eqs. 4.120, 4.124, and 4.125

$$A^2 = -\frac{\Lambda_2}{\bar{u}} (1 + \bar{u}^2) [3 C_{11} + (1 + 2 \bar{u}^2) C_{12} - (2 + \bar{u}^2) C_{21} - 3 \bar{u}^2 C_{22}]^{-1} (1 + i \omega \bar{g}_2 \bar{v} / 2 \Lambda_2) \tag{4.126}$$

This result is equal to the one given for $\psi/2$ in Ref. 1, except for the factor

$$F = 1 + \frac{1}{2\Lambda_2} i \omega \bar{g}_2 \bar{v} = 1 - \frac{1}{2\Lambda_2} \frac{\omega^2 \bar{g}_2^2}{\Omega_2^2 - \omega^2} \bar{u}$$
$$= 1 - \frac{1}{2\Lambda_2} \frac{\omega^2}{\Omega_2^2 - \Omega_1^2} (\bar{g}_1 + \bar{g}_2) \sqrt{\bar{g}_1 \bar{g}_2} \tag{4.127}$$

which is probably related to the fact that Λ_o (Eq. 4.38) is different from Λ_F (Eq. A.13)

$$\Lambda_F^2 = \Lambda_o^2 \left[1 + (g_1 + g_2)^2 \frac{\omega_o^2}{(\Omega_2^2 - \Omega_1^2)^2} \right] \tag{4.128}$$

Further analysis is needed.

Finally, consider a comparison of the coefficients β and γ used in Section 2 of Ref. 2 and the coefficients $\overline{\beta}$, $\overline{\gamma}$, β, and γ used here. Note that according to Eqs. 2.37 to 2.39 of Section 2, Ref. 2

$$\alpha = 2i\omega(1-u^2) + g_1 - g_2 u^2$$
$$\beta = \mp \frac{1}{\alpha} 2u$$
$$\gamma = \frac{1}{\alpha}\left[3C_{11} + (u^2 + 2uu^*)C_{12} - (2u^2 + uu^*)C_{21} - 3u^3u^*C_{22}\right] \tag{4.129}$$

where u is given by (see Eqs. 2.41 and 2.42 of Ref. 2)

$$u = -\sqrt{\frac{g_1}{g_2}}\; e^{-i\tan^{-1}\left(\frac{g_1+g_2}{\Omega_2^2-\Omega_1^2}\omega_F\right)} \tag{4.130}$$

Assuming

$$0 < g_1 < g_2 << 1$$

or

$$g_1 = \varepsilon \overline{g}_1$$
$$g_2 = \varepsilon \overline{g}_2 \tag{4.131}$$

Equation 4.130 yields

$$u = -\left(1 - i\varepsilon\frac{(\overline{g}_1+\overline{g}_2)\omega_F}{\Omega_2^2-\Omega_1^2} + \cdots\right)\sqrt{\overline{g}_1/\overline{g}_2}$$
$$= \overline{u} + \varepsilon\overline{v} + \cdots \tag{4.132}$$

with $\overline{u}$ given by Eq. 4.33 and $\overline{v}$ given by Eq. 4.47.

By using Eqs. 4.131 and 4.132, Eq. 4.129 yields

$$\alpha = i2\omega(1-\overline{u}^2) - \varepsilon i4\omega\overline{u}\,\overline{v}$$
$$\alpha\beta = -2(\overline{u}+\varepsilon\overline{v})$$
$$\alpha\gamma = 3\left[C_{11} + (C_{12}-C_{21})\overline{u}^2 - C_{22}\overline{u}^4\right]$$
$$+2\varepsilon\overline{v}\,\overline{u}\left[C_{12} - 2C_{21} - 3C_{22}\overline{u}^2\right] + \cdots \tag{4.133}$$

Comparing Eqs. 4.60, 4.121 and

$$\alpha = \bar{\alpha} + \varepsilon \bar{\alpha}' + \cdots$$
$$\alpha\beta = \bar{\alpha}\bar{\beta} + \varepsilon \bar{\alpha}'\bar{\beta}' + i\omega g_2 \bar{u}\bar{v} + \varepsilon i \omega \bar{g}_2 \bar{v}^2 + \cdots \tag{4.134}$$
$$\alpha\gamma = \bar{\alpha}\bar{\gamma} + \varepsilon \bar{\alpha}'\bar{\gamma}' + \cdots$$

This yields

$$\beta = \left[(\bar{\alpha}\bar{\beta} + i g_2 \bar{u}\bar{v}) + \varepsilon(\bar{\alpha}'\bar{\beta}' + i g_2 \bar{v}^2) + \cdots\right]$$
$$\times \frac{1}{\bar{\alpha}}\left[1 - \varepsilon \frac{\bar{\alpha}'}{\bar{\alpha}} + \cdots\right]$$
$$= \bar{\beta} + \frac{i g_2 \bar{u}\bar{v}}{\bar{\alpha}} + \varepsilon\left[\frac{\bar{\alpha}'}{\bar{\alpha}}(\bar{\beta}' - \bar{\beta}) + \frac{i g_2 \bar{v}^2}{\bar{\alpha}} + \cdots\right.$$
$$= \left(\bar{\beta} + \frac{i g_2 \bar{u}\bar{v}}{\bar{\alpha}}\right) + \varepsilon\left(\hat{\beta} + i \frac{1}{\bar{\alpha}} g_2 \bar{v}^2\right) + \cdots$$

$$\gamma = (\bar{\alpha}\bar{\gamma} + \varepsilon \bar{\alpha}'\bar{\gamma}')\frac{1}{\bar{\alpha}}\left(1 - \varepsilon \frac{\bar{\alpha}'}{\bar{\alpha}} + \cdots\right) \tag{4.135}$$
$$= \bar{\gamma} + \varepsilon \frac{\bar{\alpha}'}{\bar{\alpha}}(\bar{\gamma}' - \bar{\gamma}) + \cdots$$
$$= \bar{\gamma} + \varepsilon \tilde{\gamma} + \cdots$$

SECTION 5

SUMMARY AND CONCLUSIONS

5.1 Summary

In this report, the results obtained in Refs. 1 and 2 have been extended to include the following cases:

1. The effect of fifth-order nonlinear terms.
2. Flutter-buckling interaction.
3. Small damping terms.

The results of including these three aspects are discussed in this section.

5.2 Fifth-Order Nonlinear Terms

In Section 2, the analysis is extended to include fifth-order nonlinear terms. The main result of the new analysis is that the curve which gives the limit-cycle amplitude as a function of Λ can "bend".

More specifically, the third-order analysis shows that for (stabilizing) third-order nonlinear terms, there exists a stable limit cycle for Λ greater than a critical value. The fifth-order analysis shows that destabilizing nonlinear terms has the effect that the curve amplitude versus Λ, "bends" to the left, creating a second branch of the curve which represents an unstable limit cycle.

It should be noted that in order to obtain the "bending" behavior, the parameter Λ is expanded in terms of ε^2 (see Eq. 2.10). The coefficient Λ_4 (which yields the bending of the curve amplitude versus Λ) is obtained (see Subsection 2.5) by physical assumption (without any mathematical motivation). In order to verify the correctness of this assumption, the analysis is applied to a problem for which a limit-cycle solution can be easily found. The equation considered in Appendix B was "constructed" from the desired solution. The results show that the assumption yields the correct solution.

Finally, the method is compared with the "two time-scaling" technique (Refs. 8 and 9), which was modified here in order to make it sufficiently versatile.

The results obtained with the two methods (multiple-time-scaling and modified two-time-scaling) are the same (see Subsection B.6).

5.3 Flutter-Buckling Interaction

In Section 4, the analysis is extended to include the behavior of the plate in the region of interaction of flutter with buckling. At the intersection of the flutter-stability boundary with the buckling-stability boundary, the flutter frequency is equal to zero (see Section 1). Thus, disregarding the damped part, the solution does not depend upon t_o. Hence, a completely new analysis is necessary.

The result of the new analysis can be summarized as follows. In order to avoid secular terms in the third-order system, one obtains the dependence of the solution on t_1. This is given in terms of Jacobian elliptic functions. Then in order to avoid secular terms in the fourth-order system, one obtains the amplitude of the limit cycle. It should be noted that during the transient, it is impossible to eliminate all of the secular terms. By eliminating the most important ones, the variation of the amplitude with t_2 can be obtained. Further exploration is needed in order to understand the reason that the method failed in this case. A similar behavior is observed for small damping coefficients for which a possible explanation is given in Subsection 5.4.

5.4 Small Damping Coefficients

In Section 4, the analysis is reformulated by assuming that the damping coefficients are very small (of order ε). The most important results are summarized here. The second-order system yields the value of Λ which makes the system unstable. This value is (neglecting terms of order g_i) equal to the one obtained in Ref. 2. Thus, avoiding secular terms in the third-order system yields an equation for the amplitude as a function of t_2, whose solution is periodic (the period depends upon the amplitude).

Finally, by avoiding the secular terms of the fourth-order, one obtains the limit-cycle amplitude.* This amplitude is very close to the one obtained

*This is in contrast to the results obtained in Ref. 2 where a limit-cycle behavior was obtained by avoiding secular terms in the third-order system.

in Ref. 1, the only difference being a constant close to unity. However, it should be noted that during the transient, the secular terms cannot be eliminated completely. An interpretation of this is given in the following.

Consider the function given by Eqs. 2.31, 2.32, and 2.33,

$$A = |A|\, e^{i\left[-\left(\beta_I - \frac{\beta_R}{\gamma_R}\gamma_I\right)\varepsilon^2 t + \frac{\gamma_I}{\gamma_R}\ln|A| + \varphi_0\right]} \tag{5.1}$$

with

$$|A| = \left[\frac{-\gamma_R}{\beta_R} + K\, e^{\beta_R \varepsilon^2 t}\right]^{-1/2} \tag{5.2}$$

For small damping terms (of order ε) β_R and γ_R are also of order ε (see Subsection 4.9); whereas β_I and β_I are of order one. Thus setting

$$\begin{aligned} \beta_R &= \varepsilon \bar{\beta}_R \\ \gamma_R &= \varepsilon \bar{\gamma}_R \end{aligned} \tag{5.3}$$

Equations 5.1 and 5.2 can be rewritten as

$$A = |A|\, e^{i\left[-\left(\beta_I - \frac{\bar{\beta}_R}{\bar{\gamma}_R}\gamma_I\right)\varepsilon t + \frac{\gamma_I}{\varepsilon\bar{\gamma}_R}\ln|A| + \varphi_0\right]} \tag{5.4}$$

with

$$|A| = \left[\frac{-\bar{\gamma}_R}{\bar{\beta}_R} + K\, e^{\bar{\beta}_R \varepsilon^3 t}\right]^{-1/2} \tag{5.5}$$

It is known that a term such as

$$e^{i\frac{\gamma_I}{\varepsilon\bar{\gamma}_R}\ln|A|} = e^{\frac{1}{\varepsilon}f(t)} \tag{5.6}$$

has no asymptotic expansion. Thus, the presence of the term given by Eq. 5.6 in the solution could be the reason that, as mentioned above, during the transient it is impossible to avoid all the secular terms. Further exploration to verify this interpretation is needed.

5.5 Final Remarks

The results obtained here bring a new understanding of the multiple-time-scaling technique. However, the limitation mentioned in Subsections 5.3 and 5.4 need further exploration. Probably a combination of multiple-time scaling and the two-time-scaling technique (that is, a multiple-time scaling with the stretching of all the scales) might be sufficiently flexible to avoid all the secular terms. Further effort is also needed to extend the results obtained here to the case of many modes. Finally, numerical application should be done in order to compare the results of this analysis with those obtained by using a different approach.

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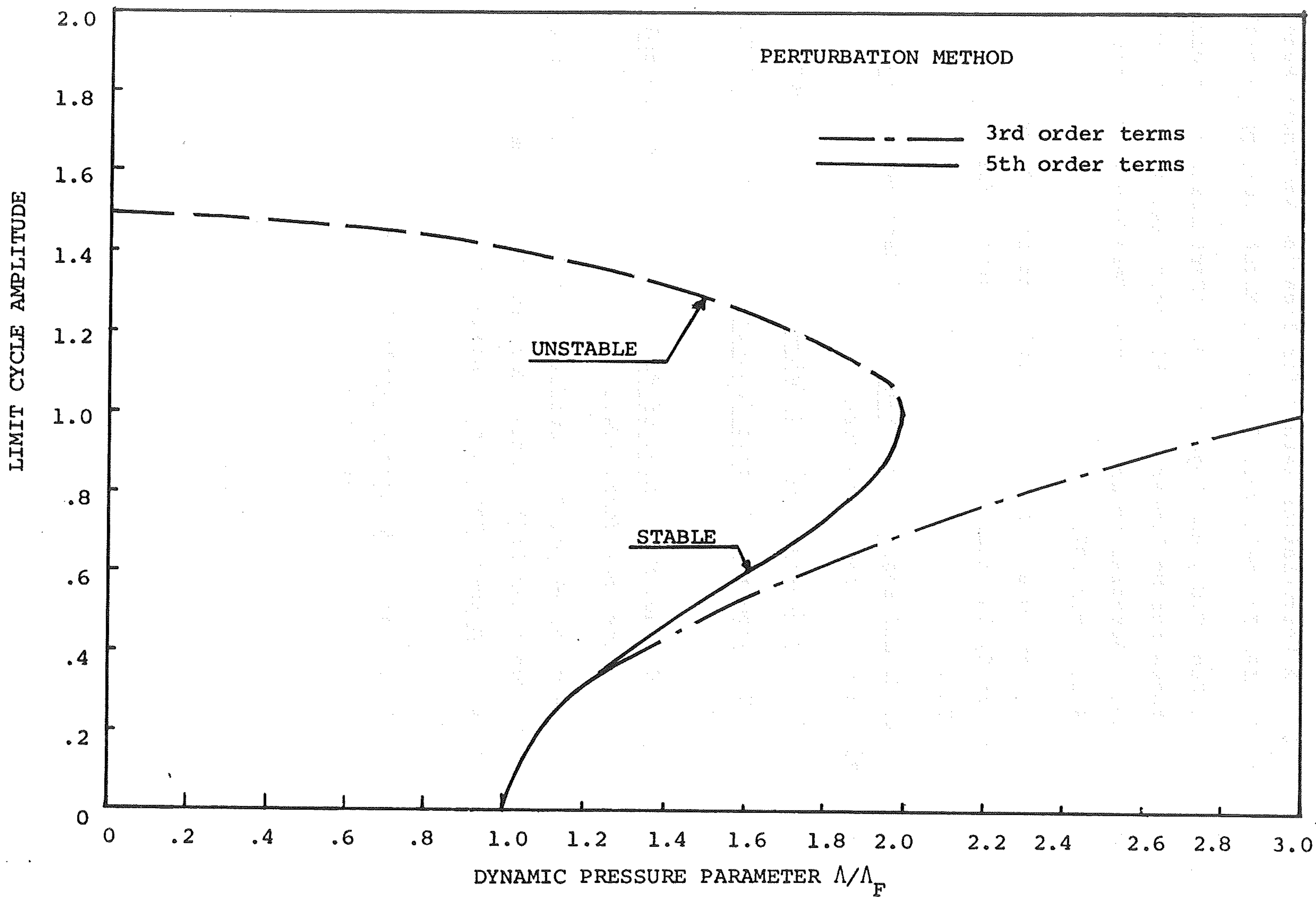


FIG. 1 EFFECT OF THE FIFTH-ORDER NONLINEARITY

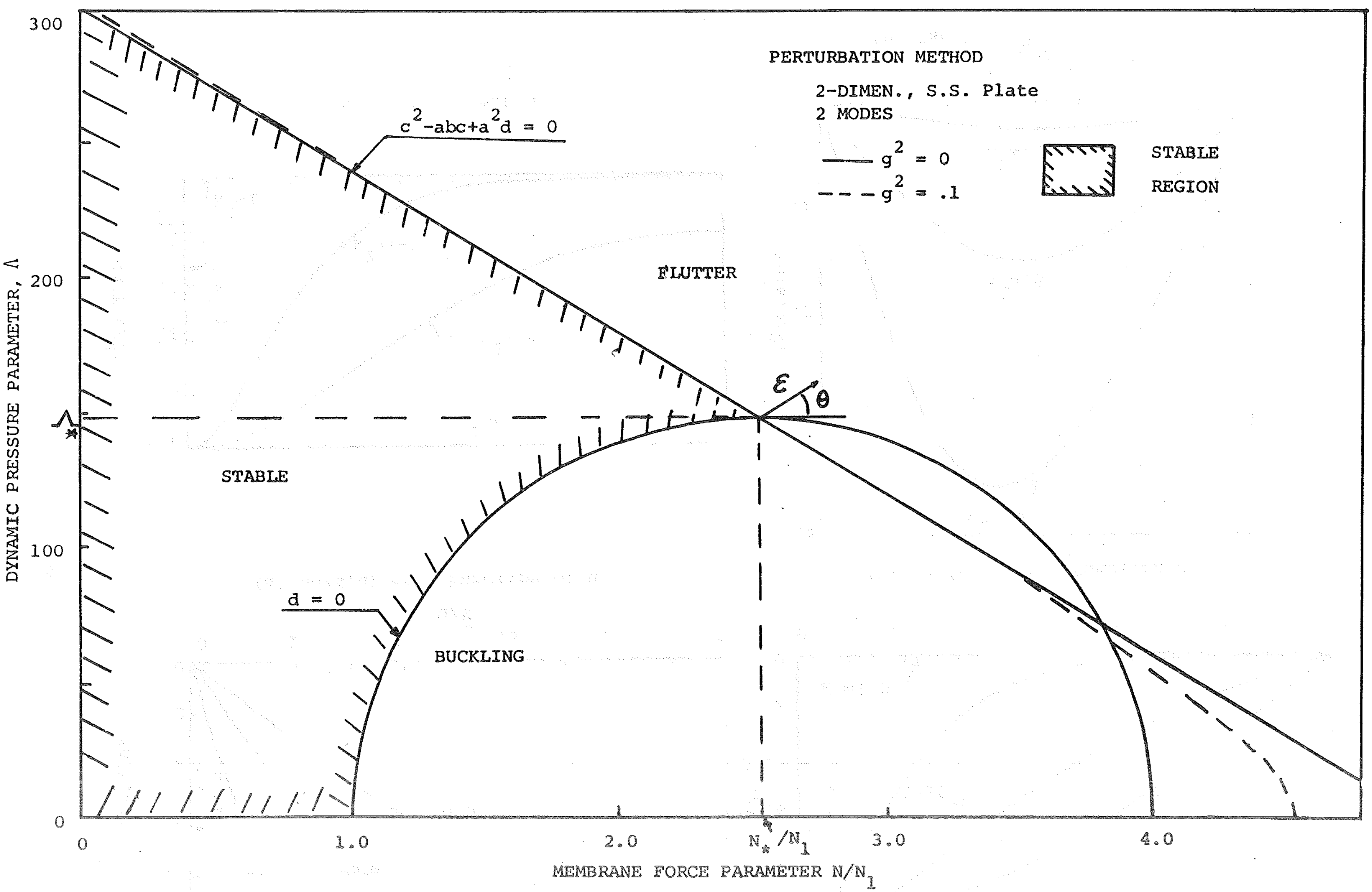
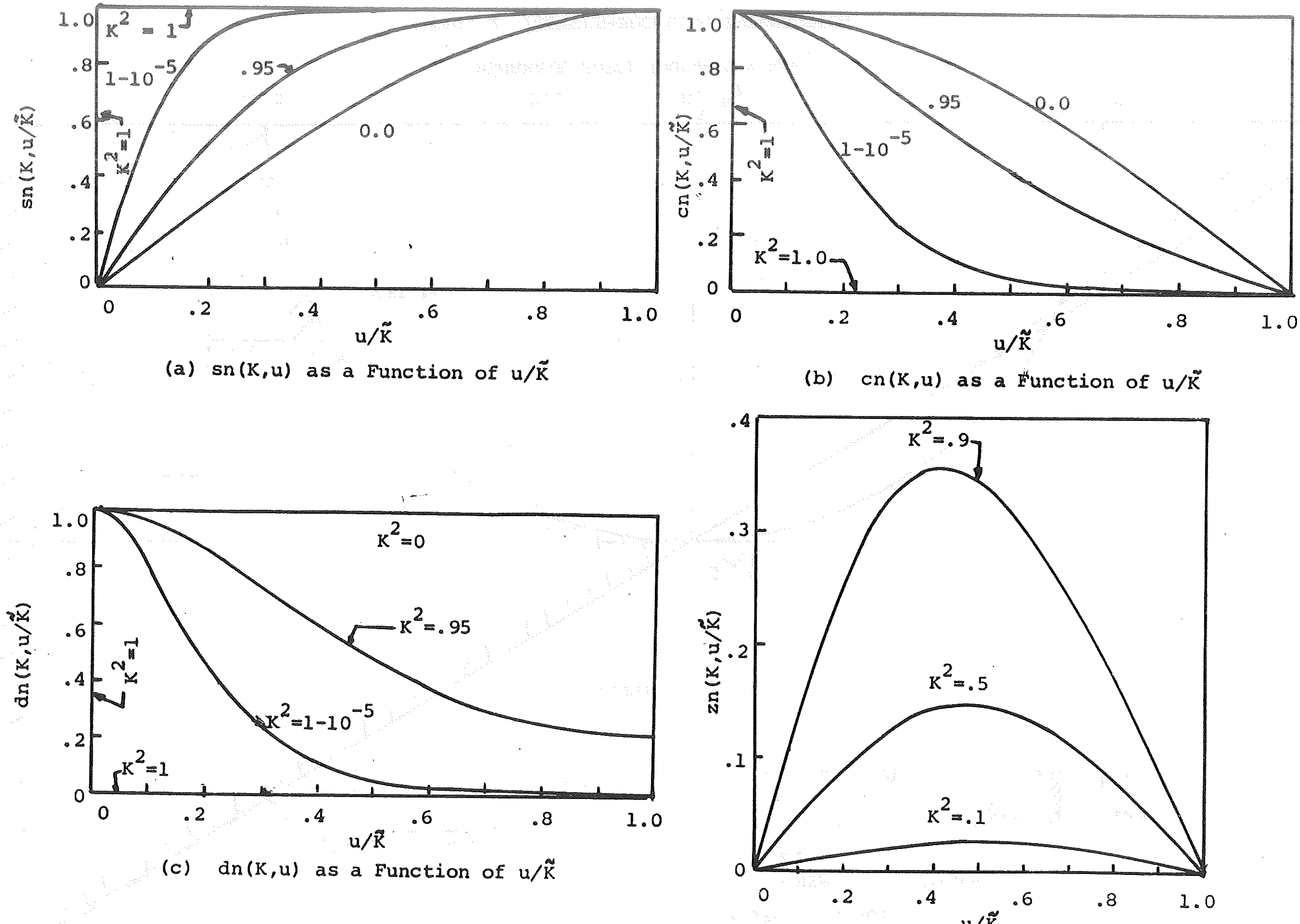


FIG. 2 FLUTTER-BUCKLING INTERACTION



(a) sn(K,u) as a Function of u/K̃

(b) cn(K,u) as a Function of u/K̃

(c) dn(K,u) as a Function of u/K̃

(d) The Jacobian Zeta Function zn(K,u) as a Function of u/K̃

FIG. 3 ELLIPTIC FUNCTIONS

APPENDIX A

THE LINEAR CASE

A.1 Introduction

As mentioned in Section 1, a clear understanding of the linearized panel-flutter problem is necessary in order to study the nonlinear panel flutter. The results of the linearized analysis are particularly important for the formulation of the flutter-buckling interaction. In this appendix, the linearized problem is studied in detail.

The linearized equation of the N-mode panel flutter are obtained from Eq. 1.3 by dropping the nonlinear terms:

$$\ddot{W}_n + g_n \dot{W}_n + \Omega_n^2 W_n + \lambda \sum_{p=1}^{N} e_{np} W_p = 0 \tag{A.1}$$

In particular, for the two-mode case, Eq. 1.8 yields

$$\begin{aligned} &\ddot{W}_1 + g_1 \dot{W}_1 + \Omega_1^2 W_1 - \Lambda W_2 = 0 \\ &\ddot{W}_2 + g_2 \ddot{W}_2 + \Omega_2^2 W_2 + \Lambda W_1 = 0 \end{aligned} \tag{A.2}$$

with Λ given by Eq. 1.10. The region of stability of Eq. A.2 is determined in Subsections A.2, A.3, and A.4. Particular emphasis is given to the intersection of the flutter boundary with the buckling boundary. The results obtained are applied in Subsection A.5 to the case of an infinite simply-supported plate for which (see Ref. 2, Appendix A)

$$\begin{aligned} &\Omega_n^2 = \Omega_{n,0}^2 \left(1 - \frac{N}{N_n}\right) \\ &g_n = G + n^4 F \end{aligned} \tag{A.3}$$

where N is the applied membrane force and

$$\Omega_{n,0} = n^4 \pi^4$$

$$N_n = \frac{n^2 \pi^2 D}{a^2}$$

$$G = (\lambda \mu / M)^{1/2}$$

$$F = g_E \pi^4 \tag{A.4}$$

The solution for Eq. A.2 on the flutter boundary and at the flutter-buckling-intersection point is discussed in Subsections A.6 and A.7, respectively. Furthermore, some generalizations are discussed in Subsection A.8.

A.2 General Formulation - Buckling and Flutter

As is well known, the solution for Eq. A.2 is a linear combination of particular solutions of the form

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} e^{pt} \tag{A.5}$$

Combining Eqs. A.2 and A.5 yields the algebraic system connected to the differential system given by Eq. A.2. The algebraic system, written in matrix form, is given by

$$\begin{bmatrix} p^2 + g_1 p + \Omega_1^2 & -\Lambda \\ \Lambda & p^2 + g_2 p + \Omega_2^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0 \tag{A.6}$$

The characteristic values of p are obtained by solving the characteristic equation. The characteristic equation of this system is obtained by setting the determinant of the coefficients equal to zero, which yields

$$p^4 + a p^3 + b p^2 + c p + d = 0 \tag{A.7}$$

with

$$a = g_1 + g_2$$
$$b = \Omega_1^2 + \Omega_2^2 + g_1 g_2$$
$$c = g_1 \Omega_2^2 + g_2 \Omega_1^2$$
$$d = \Omega_1^2 \Omega_2^2 + \Lambda^2 \qquad \text{(A.8)}$$

The stability boundary is the line separating a region of stable solution (real p < 0) and a region of unstable solution (real p > 0); that is the line on which real p = 0. Thus, the stability boundary is obtained by setting $p = i\omega$ and equating both real and imaginary parts of Eq. A.7 to zero. This yields

$$\omega^4 - (\Omega_1^2 + \Omega_2^2 + g_1 g_2)\,\omega^2 + (\Omega_1^2 \Omega_2^2 + \Lambda^2) = 0 \qquad \text{(A.9)}$$

$$\omega\left[(g_1 + g_2)\,\omega^2 - (g_1 \Omega_2^2 + g_2 \Omega_1^2)\right] = 0 \qquad \text{(A.10)}$$

The second equation yields two possibilities: assuming that $\omega \neq 0$ yields

$$\omega_F^2 = \frac{g_1 \Omega_2^2 + g_2 \Omega_1^2}{g_1 + g_2} \qquad \text{(A.11)}$$

and the first

$$\begin{aligned} \Lambda_F^2 &= -\omega_F^4 + (\Omega_1^2 + \Omega_2^2 + g_1 g_2)\,\omega_F^2 - \Omega_1^2 \Omega_2^2 \\ &= -(\omega_F^2 - \Omega_1^2)(\omega_F^2 - \Omega_2^2) + g_1 g_2 \omega_F^2 \\ &= -\left(\frac{g_1 \Omega_2^2 + g_2 \Omega_1^2}{g_1 + g_2} - \Omega_1^2\right)\left(\frac{g_1 \Omega_2^2 + g_2 \Omega_1^2}{g_1 + g_2} - \Omega_2^2\right) + g_1 g_2 \omega_F^2 \end{aligned} \qquad \text{(A.12)}$$

or

$$\Lambda_F^2 = \frac{g_1 g_2}{(g_1 + g_2)^2}(\Omega_2^2 - \Omega_1^2)^2 + g_1 g_2 \omega_F^2 \qquad \text{(A.13)}$$

The subscript F stands for flutter*. Equation A.12 gives the flutter value for the dynamic pressure parameter Λ, and Eq. A.11 gives the flutter frequency.

Consider next, the second possibility in Eq. A.10

$$\omega_B = 0 \tag{A.14}$$

Combining with Eq. A.9 yields

$$\Lambda_B^2 = -\Omega_1^2 \Omega_2^2 \tag{A.15}$$

The subscript B has been used because Eq. A.15 represents the buckling condition**. The buckling frequency is zero.

A.3 General Study of Stability

The conditions for the stability of the characteristic equation given by Eq. A.7 are: $a > 0$, $b > 0$, $c > 0$, $d > 0$, and

$$c^2 - abc + a^2 d < 0 \tag{A.16}$$

The first condition yields

$$g_1 + g_2 > 0 \tag{A.17}$$

In the following, it is assumed that $g_k > 0$ so that this condition is always satisfied. The second, third, and fourth conditions yield

$$\Omega_1^2 + \Omega_2^2 + g_1 g_2 > 0 \tag{A.18}$$

$$g_1 \Omega_2^2 + g_2 \Omega_1^2 > 0 \tag{A.19}$$

$$\Omega_1^2 \Omega_2^2 + \Lambda^2 > 0 \tag{A.20}$$

---

*Flutter is an instability of vibration with growing amplitude; that is $p = \beta \pm i\omega$ with $\beta > 0$. The flutter boundary is defined by $\beta = 0$.

**Buckling is an instability with exponential growth;that is p is real and positive. The buckling boundary is defined by $p = 0$.

Finally, Eq. A.16 may be rewritten as

$$\left(\frac{c}{a}\right)^2 - b\left(\frac{c}{a}\right) + d < 0 \tag{A.21}$$

which yields

$$\left(\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2}\right)^2 - \left(\Omega_1^2+\Omega_2^2+g_1g_2\right)\left(\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2}\right) + \Lambda^2 + \Omega_1^2\Omega_2^2 < 0 \tag{A.22}$$

or

$$\begin{aligned}\Lambda^2 &< \left(\Omega_1^2+\Omega_2^2+g_1g_2\right)\left(\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2}\right) - \Omega_1^2\Omega_2^2 - \left(\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2}\right)^2 \\ &= -\left(\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2} - \Omega_1^2\right)\left(\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2} - \Omega_2^2\right) \\ &\quad + g_1g_2\,\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2}\end{aligned} \tag{A.23}$$

which yields

$$\Lambda^2 < \frac{g_1g_2}{(g_1+g_2)^2}\left(\Omega_2^2-\Omega_1^2\right)^2 + g_1g_2\,\frac{g_1\Omega_2^2+g_2\Omega_1^2}{g_1+g_2} \tag{A.24}$$

It should be noted that this condition corresponds to the flutter condition as is seen by comparing Eqs. A.13 and A.24. Similarly, comparing Eqs. A.15 and A.20 shows that the condition $d > 0$ corresponds to the buckling condition. Finally, the condition $c > 0$ corresponds to the condition $\omega_F^2 > 0$ (compare Eqs. A.26 and A.19).

In order to determine the region of stability, it is convenient to find the conditions under which $\Lambda_F = \Lambda_B$. The remainder of Subsection A.3 is devoted to finding these conditions. Using Eqs. A.12 and A.15, yields*

$$\Lambda_F^2 - \Lambda_B^2 = \omega_F^2\left(\Omega_2^2+\Omega_1^2+g_1g_2-\omega_F^2\right) = 0 \tag{A.25}$$

*This corresponds to satisfying simultaneously the conditions $c^2 = abc - a^2d = 0$, and $d = 0$, which implies that $c(c - ab) = 0$.

Thus, there exist two intersecting points defined by

$$\omega_F^2 = \frac{g_2\Omega_1^2 + g_1\Omega_2^2}{g_1 + g_2} = 0 \qquad \text{(intersection 1)} \qquad (A.26)$$

and

$$\frac{g_1\Omega_1^2 + g_2\Omega_2^2}{g_1 + g_2} + g_1 g_2 = 0 \qquad \text{(intersection 2)} \qquad (A.27)$$

A.4 The Region of Stability in the Λ,N Plane

In the preceding subsection, the conditions for the stability of the characteristic equation were derived. These conditions are:

$b > 0$, or Eq. A.18

$$\Omega_1^2 + \Omega_2^2 + g_1 g_2 > 0 \qquad \text{(condition b)} \qquad (A.28)$$

$c > 0$, or Eq. A.19,

$$g_1\Omega_2^2 + g_2\Omega_1^2 > 0 \qquad \text{(condition c)} \qquad (A.29)$$

$d > 0$, or Eq. A.20

$$\Lambda^2 > \Lambda_B^2 = -\Omega_1^2\Omega_2^2 \qquad \text{(buckling condition)} \qquad (A.30)$$

$c^2 - abc + a^2 d < 0$, or Eq. A.24

$$\Lambda^2 < \Lambda_F^2 = \frac{g_1 g_2}{g_1 + g_2}\left(\Omega_2^2 - \Omega_1^2\right)^2 + g_1 g_2 \frac{g_1\Omega_2^2 + g_2\Omega_1^2}{g_1 + g_2} \qquad \text{(flutter condition)} \qquad (A.31)$$

Finally, the conditions for $\Lambda_F^2 = \Lambda_B^2$ are either Eq. A.26

$$g_2\Omega_1^2 + g_1\Omega_2^2 = 0 \qquad \text{(intersection 1)} \qquad (A.32)$$

or Eq. A.27

$$\frac{g_1\Omega_1^2 + g_2\Omega_2^2}{g_1 + g_2} + g_1 g_2 = 0 \qquad \text{(intersection 2)} \qquad (A.33)$$

In the following, these equations are rewritten by expressing Ω_1^2 and Ω_2^2 in terms of the membrane-compression force N, according to Eq. A.2,

$$\Omega_1^2 = \Omega_{10}^2 \left(1 - \frac{N}{N_1}\right)$$

$$\Omega_2^2 = \Omega_{20}^2 \left(1 - \frac{N}{N_2}\right) \tag{A.34}$$

Consider first, condition b and condition c. Combining Eqs. A.28, A.29, and A.34 yields

$$b = \Omega_{10}^2 \left(1 - \frac{N}{N_1}\right) + \Omega_{20}^2 \left(1 - \frac{N}{N_2}\right) + g_1 g_2 > 0 \tag{A.35}$$

$$c = g_2 \Omega_{10}^2 \left(1 - \frac{N}{N_1}\right) + g_1 \Omega_{20}^2 \left(1 - \frac{N}{N_2}\right) > 0 \tag{A.36}$$

or

$$N < N_b \tag{A.37}$$

and

$$N < N_c \tag{A.38}$$

with

$$N_b = \frac{\Omega_{10}^2 + \Omega_{20}^2 + g_1 g_2}{\Omega_{10}^2 + \Omega_{20}^2 \frac{N_1}{N_2}} N_1 = \frac{1 + R_\Omega + g^2}{1 + R_N} N_1 \tag{A.39}$$

$$N_c = \frac{g_2 \Omega_{10}^2 + g_1 \Omega_{20}^2}{g_2 \Omega_{10}^2 + g_1 \Omega_{20}^2 \frac{N_1}{N_2}} N_1 = \frac{\theta + R_\Omega}{\theta + R_N} N_1 \tag{A.40}$$

where the following definitions have been used:

$$R_\Omega = \frac{\Omega_{20}^2}{\Omega_{10}^2}\ ; \quad R_N = \frac{\Omega_{20}^2}{\Omega_{10}^2} \frac{N_1}{N_2}\ ; \quad \theta = \frac{g_2}{g_1}\ ; \quad g^2 = \frac{g_1 g_2}{\Omega_{10}^2} \tag{A.41}$$

Consider next, the buckling and the flutter conditions. Combining Eqs. A.30 and A.34 yields

$$\Lambda^2 > \Lambda_B^2 = -\Omega_{10}^2\Omega_{20}^2\left(1-\frac{N}{N_1}\right)\left(1-\frac{N}{N_2}\right)$$

$$= -\left(1-\frac{N}{N_1}\right)\left(R_\Omega - R_N\frac{N}{N_1}\right)\Omega_{10}^4 \tag{A.42}$$

Similarly, combining Eqs. A.31 and A.34 yields

$$\Lambda^2 < \Lambda_F^2 = \frac{g_1 g_2}{(g_1+g_2)^2}\left[\Omega_{10}^2\left(1-\frac{N}{N_1}\right) - \Omega_{20}^2\left(1-\frac{N}{N_2}\right)\right]^2$$

$$+ g_1 g_2 \frac{g_1\Omega_{20}^2\left(1-\frac{N}{N_2}\right) + g_2\Omega_{10}^2\left(1-\frac{N}{N_1}\right)}{g_1+g_2}$$

$$= \left\{\frac{\theta}{(\theta+1)^2}\left[(1-R_\Omega)-(1-R_N)\frac{N}{N_1}\right]^2\right.$$

$$\left. + \frac{g^2}{1+\theta}\left[(\theta+R_\Omega)-(\theta+R_N)\frac{N}{N_1}\right]\right\}\Omega_{10}^4$$

(A.43)

Finally, the two intersection conditions define the two values for N ($N^{(1)}$ and $N^{(2)}$ for intersections 1 and 2, respectively). Since the equation for intersection 1, Eq. A.32, is equal to the equation c = 0 (see Eq. A.36), then necessarily

$$N^{(1)} = N_c \tag{A.44}$$

with N_c given by Eq. A.40. In order to find $N^{(2)}$ combine Eqs. A.33 and A.34

$$g_1\Omega_{10}^2\left[1-\frac{N^{(2)}}{N_1}\right] + g_2\Omega_{20}^2\left[1-\frac{N^{(2)}}{N_2}\right] + g_1 g_2(g_1+g_2) = 0 \tag{A.45}$$

or

$$N^{(2)} = \frac{g_1\Omega_{1b}^2 + g_2\Omega_{2b}^2 + g_1 g_2 (g_1 + g_2)}{g_1\Omega_{1b}^2 + g_2\Omega_{2b}^2 \frac{N_1}{N_2}} N_1$$

$$= \frac{1 + \theta R_\Omega + g^2(1+\theta)}{1 + \theta R_N} N_1 \tag{A.46}$$

A.5 The Region of Stability in the Λ,N Plane for an Infinite Plate

For an infinite plate, according to Eq. A.4, one has

$$R_\Omega = \frac{\Omega_{2b}^2}{\Omega_{1b}^2} = 16 \qquad R_N = \frac{\Omega_{2b}^2}{\Omega_{1b}^2}\frac{N_1}{N_2} = 4 \tag{A.47}$$

whereas

$$\theta = \frac{g_2}{g_1} = \frac{G + 16F}{G + F}$$

ranges from 1 (F = 0) to 16 (G = 0). Thus, Eqs. A.39, A.40, A.44, and A.45 become

$$\frac{N_b}{N_1} = \frac{17}{5} + \frac{1}{5}g^2$$

$$\frac{N^{(1)}}{N_1} = \frac{N_c}{N_1} = \frac{\theta + 16}{\theta + 4}$$

$$\frac{N^{(2)}}{N_1} = \frac{16\theta + 1}{4\theta + 1} + g^2\frac{\theta + 1}{4\theta + 1} \tag{A.48}$$

Similarly, Eqs. A.42 and A.43 yield,

$$\frac{\Lambda_B^2}{\pi^8} = \left(16 - 20\frac{N}{N_1} + 4\frac{N^2}{N_1^2}\right)$$

$$\frac{\Lambda_F^2}{\pi^8} = \frac{\theta}{(\theta+1)^2}\left(15 - \frac{3N}{N_1}\right)^2 + g^2\,\frac{\theta\left(1-\frac{N}{N_1}\right) + \left(16 - 4\frac{N}{N_1}\right)}{\theta + 1} \tag{A.49}$$

The quantities N_b, N_c, $N^{(1)}$, and $N^{(2)}$, versus θ are plotted in Fig. A.1 for $g^2=0$ and $g^2 = .1$. This figure shows that the influence of $g^2 = (g_1 g_2)/\pi^4$ is negligible, whereas the influence of $\theta = g_2/g_1$ is strongly significant. In other words, the ratio of the two damping coefficients is important, whereas the influence of g^2 itself is generally negligible.*

By examining Fig. A.1, it can be noted that in the range $\theta = 1$ to $\theta = 16$**:

(a) condition c is more restrictive than condition b, $N_c < N_b$

(b) intersection 1 corresponds to a lower value of N than intersection 2, $N^{(1)} < N^{(2)}$

(c) the fact that $N^{(1)} = N_c$ implies that at the intersection of the flutter boundary with the buckling boundary at $N = N^{(1)}$, the flutter frequency, which goes to zero with the coefficient c, is equal to zero, since $N = N_c$

The stability region in the λ,N plane is shown (for g = 0 and g = .1) in Figs. A.2, A.3, and A.4 for θ = 1, 4, and 16, respectively.

A.6 Solution for the Flutter Boundary

In the preceding subsections, it was shown that for

$$\Lambda^2 = \Lambda_F^2 = \frac{g_1 g_2}{(g_1+g_2)^2}\left(\Omega_2^2 - \Omega_1^2\right)^2 + g_1 g_2\,\frac{g_1\Omega_2^2 + g_2\Omega_1^2}{g_1 + g_2} \tag{A.50}$$

---

*In the usual physical problems, $g^2 = (g_1 g_2)/\Omega_{10}^2$ is less than 1/10.

**It should be noted that this result is true only for $\theta > 1$ (i.e., $g_2 > g_1$ and $g^2 > 0$). The case $\theta < 1$, as well as the case $\theta = 1$, $g^2 = 0$, is discussed in Subsection A.8.

and

$$\omega_F^2 = \frac{g_1\Omega_2^2 + g_2\Omega_1^2}{g_1 + g_2} > 0 \qquad (N < N_c) \tag{A.51}$$

the linear system is on the flutter boundary (that is, between damped and growing oscillations). In this section, the solution to the linear system on the flutter boundary is described in detail.

As shown in Subsection A.2, if $\lambda^2 = \lambda_F^2$, the characteristic equation (Eq. A.6) has two imaginary roots given by

$$p = \pm i\,\omega_F \tag{A.52}$$

The other two roots can be found by dividing the characteristic equation by $p^2 + \omega_F^2$, which yields*

$$p^2 + (g_1+g_2)\,p + \frac{g_1\Omega_1^2 + g_2\Omega_2^2}{g_1 + g_2} + g_1 g_2 = 0 \tag{A.53}$$

Thus, the second pair of roots is given by

$$p = -\frac{g_1+g_2}{2} \pm i\sqrt{\frac{g_1\Omega_1^2 + g_2\Omega_2^2}{g_1+g_2} + g_1 g_2 - \left(\frac{g_1+g_2}{2}\right)^2}$$

$$= -\frac{g_1+g_2}{2} \pm i\sqrt{\frac{g_1\Omega_1^2 + g_2\Omega_2^2}{g_1+g_2} - \left(\frac{g_2-g_1}{2}\right)^2} \tag{A.54}$$

---

*Note that

$$(p^2+\omega_F^2)\left[p^2 + (g_1+g_2)\,p + (\Omega_1^2+\Omega_2^2+g_1g_2-\omega_F^2)\right]$$

$$= p^4 + (g_1+g_2)p^3 + (\Omega_1^2+\Omega_2^2+g_1g_2)p^2 + (g_1+g_2)\omega_F^2 p + \omega_F^2(\Omega_1^2+\Omega_2^2+g_1g_2-\omega_F^2)$$

and (using Eqs. A.11 and A.13)

$$\Omega_1^2+\Omega_2^2+g_1g_2-\omega_F^2 = \frac{g_1\Omega_1^2+g_2\Omega_2^2}{g_1+g_2} + g_1g_2$$

$$\omega_F^2(\Omega_1^2+\Omega_2^2+g_1g_2-\omega_F^2) = \Lambda_F^2 + \Omega_1^2\Omega_2^2$$

This is the same as saying that

$$\left(p^2+\frac{c}{a}\right)\left[p^2+ap-\left(b-\frac{c}{a}\right)\right] = p^4 + ap^3 + bp^2 + cp + d$$

since

$$c^2 - abc + a^2 d = 0$$

These form a complex conjugate pair of roots with a negative real part. These roots are complex conjugates if

$$\frac{g_1 \Omega_1^2 + g_2 \Omega_2^2}{g_1 + g_2} - \left(\frac{g_1 - g_2}{2}\right)^2 > 0 \tag{A.55}$$

i.e., if

$$N > N' = \frac{g_1 \Omega_{10}^2 + g_2 \Omega_{20}^2 - \left(\frac{g_1 - g_2}{2}\right)^2 (g_1 + g_2)}{g_1 \Omega_{10}^2 + g_2 \frac{N_1}{N_2} \Omega_{20}^2} N_1$$

$$= \frac{1 + \theta R_\Omega - g^2 \left(\frac{1+\theta}{4\theta}\right)(1-\theta)^2}{1 + \theta R_N} N_1 \tag{A.56}$$

In particular, for a simply-supported plate,

$$N' = \frac{1 + 16\theta - \frac{g^2}{4} \frac{1+\theta}{\theta} (1-\theta)^2}{1 + 4\theta} N_1 \tag{A.57}$$

It can be noted that for $g^2 < .1$ and $1 < \theta \leq 16$, N^1 is always greater than N_c; but $N^1 = N_c$ for $\theta = 1$. Since $N < N_c$ (see Eq. A.51) the roots given by Eq. A.54 are always a complex conjugate pair of roots with a negative real part.

Thus, on the flutter boundary, the characteristic equation has two imaginary roots $p = \pm i\omega_F$ and two damped roots; disregarding the damped part, the solution can be written as

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \{U\} A e^{i\omega_F t} + \{U^*\} A^* e^{-i\omega_F t} \tag{A.58}$$

where A is an arbitrary complex constant and U is the eigenvector of Eq. A.6 with $p = i\omega_F$ and is given by

$$\{U\} = \begin{Bmatrix} 1 \\ u \end{Bmatrix} \tag{A.59}$$

with*

$$U = \frac{W_2}{W_1} = -\sqrt{\frac{g_1}{g_2}}\, e^{i\chi} \tag{A.60}$$

where

$$\chi = -\tan^{-1}\left(\frac{g_1 + g_2}{\Omega_2^2 - \Omega_1^2}\,\omega_F\right) < 0 \tag{A.61}$$

Finally, it is convenient to mention that the left eigenvector $\lfloor U^L \rfloor$, which is needed in the nonlinear analysis,(see Subsection 1.3) is given by

$$\lfloor U^L \rfloor = \lfloor 1, -U \rfloor \tag{A.62}$$

A.7 The Solution at the Interaction Point

In the preceding subsections, it was shown that the stability region is limited by the conditions

$$\Lambda^2 \leq \Lambda_F^2(N) \qquad \text{(flutter)} \tag{A.63a}$$

$$\Lambda^2 \leq \Lambda_B^2(N) \qquad \text{(buckling)} \tag{A.63b}$$

There exists an interaction point (which is part of the boundary of the

---

*In fact

$$U = \frac{W_1}{W_2} = \frac{-\omega_F^2 + i g_1 \omega_F + \Omega_1^2}{\Lambda_F} = \frac{1}{\Lambda_F}\left[\frac{-g_1}{g_1+g_2}(\Omega_2^2 - \Omega_1^2) + i g_1 \omega_F\right]$$

$$= \frac{-1}{\Lambda_F}\left[\frac{g_1^2}{(g_1+g_2)^2}(\Omega_2^2-\Omega_1^2)^2 + g_1^2\omega_F^2\right]^{1/2} e^{-i\tan^{-1}\frac{g_1+g_2}{\Omega_2^2-\Omega_1^2}\omega_F}$$

$$= -\sqrt{\frac{g_1}{g_2}}\, e^{i\chi}$$

stability region) where $\Lambda^2 = \Lambda_F^2$ and $\Lambda^2 = \Lambda_B^2$, simultaneously. These conditions are equivalent to the conditions

$$c^2 - abc + a^2 d = 0 \quad , \quad d = 0 \tag{A.64}$$

on the coefficients of the characteristic equation. Equation A.64 implies that at the interaction point, either c = ab, or c = 0. In Subsection A.5, it was shown (for an infinite plate) that at the interaction point c = 0. The general conditions under which c = 0 at the interaction point are discussed in Subsection A.8. In the following, the solution of the linear system at the interaction point is studied in detail. The conditions d = 0 and c = 0 correspond to (see Eqs. A.32 and A.44)

$$N = N_* = N^{(1)} = \frac{g_2 \Omega_{10}^2 + g_1 \Omega_{20}^2}{g_2 \Omega_{10}^2 + g_1 \Omega_{20}^2 \frac{N_1}{N_2}}$$

$$\Lambda = \Lambda_* = -\Omega_2^2 \Omega_1^2 = -\Omega_{10}^2 \Omega_{20}^2 \left(1 - \frac{N^{(1)}}{N_1}\right)\left(1 - \frac{N^{(1)}}{N_2}\right) \tag{A.65}$$

Since c = d = 0, Eq. A.7 reduces to

$$p^4 + (g_1 + g_2) p^3 + (\Omega_1^2 + \Omega_2^2 + g_1 g_2) p^2 = 0 \tag{A.66}$$

which has a double root p = 0 and a pair of roots given by

$$p = -\frac{g_1 + g_2}{2} \pm i \sqrt{\Omega_1^2 + \Omega_2^2 - \left(\frac{g_1 - g_2}{2}\right)^2} \tag{A.67}$$

These roots are complex conjugate for $\theta \neq 1$, real and double for $\theta = 1$;* in

---

*These roots are complex conjugates if

$$\Omega_1^2 + \Omega_2^2 - \left(\frac{g_1 - g_2}{2}\right)^2 > 0$$

That is, if

$$N > N'' = \frac{\Omega_{10}^2 + \Omega_{20}^2 - \left(\frac{g_1 - g_2}{2}\right)^2}{\Omega_{10}^2 + \frac{N_1}{N_2}\Omega_{20}^2} N_1 = \frac{1 + R_\Omega - g^2 \frac{(1-\theta)^2}{4\theta}}{1 + R_N} N_1$$

In particular, for a simply-supported plate

$$N'' = \frac{1}{5}\left[17 - g^2 \frac{(1-\theta)^2}{4\theta}\right] N_1$$

Note that for $1 < \theta \leq 16$; $N'' > N_c$; thus, in this case the roots are complex conjugates. But $N'' = N_c$ for $\theta = 1$. In this case the radical is equal to zero and the roots reduce to a double real negative root given by $p = -2g_1$ (note that $g_1 = \theta g_2 = g_2$.

either case, they have negative real parts. Disregarding the damped part, the solution can be written as

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = A\{U\} + A_1\left(\{U\}t + \{U_1\}\right) \tag{A.68}$$

where A and A_1 are two arbitrary constants and U and U_1 are given by*

$$\{U\} = \begin{Bmatrix} 1 \\ u \end{Bmatrix} \qquad \{U_1\} = \begin{Bmatrix} 0 \\ u_1 \end{Bmatrix} \tag{A.69}$$

with

$$u = -\sqrt{\frac{g_1}{g_2}} \qquad u_1 = \frac{\sqrt{g_1 g_2}}{\Omega_2^2} = -\frac{g_2 u}{\Omega_2^2} = \frac{g_1}{\Lambda} \tag{A.70}$$

Finally, it is convenient to mention that the left eigenvector $\lfloor U^L \rfloor$, which is needed in the nonlinear analysis is given by (see Subsection 1.3)

$$\lfloor U^L \rfloor = \lfloor 1, -u \rfloor \tag{A.71}$$

*In fact, setting $\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ u \end{Bmatrix} t + \begin{Bmatrix} 0 \\ u_1 \end{Bmatrix}$ in Eq. A.2 yields

yields

$$0 + g_1(1+0) + \Omega_1^2(t+0) - \Lambda(tu + u_1) = 0$$

$$0 + g_2(u+0) + \Omega_2^2(tu + u_1) + \Lambda(t+0) = 0$$

or (note that $g_1\Omega_2^2 + g_2\Omega_1^2 = 0$ and $\Lambda = -\Omega_1^2\Omega_2^2 = \frac{g_1}{g_2}\Omega_2^4$)

$$u = \frac{\Omega_1^2}{\Lambda} = -\frac{\Lambda}{\Omega_2^2} = -\left|\frac{\Omega_1}{\Omega_2}\right| = -\sqrt{\frac{g_1}{g_2}}$$

$$u_1 = -\frac{g_2}{\Omega_2^2}u = \frac{g_1}{\Lambda} = g_1\sqrt{\frac{g_2}{g_1\Omega_2^4}} = \sqrt{g_1 g_2}\frac{1}{\Omega_2^2} = g_1\frac{1}{|\Omega_1\Omega_2|}$$

A.8 General Remarks

Some of the results obtained in the preceding subsections are valid only for the simply-supported plate. In this subsection, the results are extended to the general case under the conditions that Ω_1^2 and Ω_2^2 are given by Eq. A.3 with

$$N_1 > N_2 \tag{A.72}$$

where N_1 is the first buckling load (Euler load) and N_2 is the second one, and

$$g^2 > 0 \tag{A.73}$$

It may be noted that the quantities R_N, R_Ω and θ, given by Eq. A.41 are positive and furthermore, that according to Eq. A.72

$$R_\Omega > R_N$$

Consider Eqs. A.39, A.40, A.44, and A.46

$$\frac{N_c}{N_1} = \frac{N^{(1)}}{N_1} = \frac{\theta + R_\Omega}{\theta + R_N}$$

$$\frac{N^{(2)}}{N_1} = \frac{R_\Omega \theta + 1}{R_N \theta + 1} + g^2 \frac{\theta + 1}{R_N \theta + 1}$$

$$\frac{N_b}{N_1} = \frac{1 + R_\Omega}{1 + R_N} + g^2 \frac{1}{1 + R_N} \tag{A.74}$$

Note that

$$\frac{N^{(1)}}{N_1} = \frac{N_2}{N_1} = \frac{N_b}{N_1} = \frac{N_c}{N_1} = \frac{1 + R_\Omega}{1 + R_N} + g^2 \frac{1}{1 + R_N} \tag{A.75}$$

for

$$\theta = \theta_* \equiv \frac{1 - \frac{R_N}{R_\Omega - R_N} g^2}{1 + \frac{1}{R_\Omega - R_N} g^2} < 1 \tag{A.76}$$

Note also that

$$N^{(1)} = N_c < N_b < N^{(2)} \qquad \text{for} \quad \theta > \theta_* \tag{A.77}$$

whereas

$$N^{(1)} = N_c > N_b > N^{(2)} \qquad \text{for} \quad \theta < \theta_* \tag{A.78}$$

Thus, the interaction point is defined by

$$N = N^{(1)} \qquad \text{for} \quad \theta > \theta_* \tag{A.79}$$

(which implies also that $N = N_c$ and thus, the flutter frequency is equal to zero) or

$$N = N^{(2)} \qquad \text{for} \quad \theta < \theta_*$$

For this second case, the roots are given by $p = 0$, $p = \pm\, \omega_F$, and the fourth root is real and negative. Finally, $\theta = \theta_*$ yields $b = c = d = 0$; thus, in this case there is a triple root $p = 0$, and a negative real root $p = -\, a$.

It should be noted that these last two cases imply

$$0 < \theta \leq \theta_* < 1 \tag{A.80}$$

which is equivalent to

$$g_1 > g_2 > 0 \tag{A.81}$$

This condition is never satisfied with the kind of damping considered here.

Finally, it should be noted that if $N^{(1)} < 1$, there is no intersection point since $N^{(1)} < 1$ corresponds to imaginary values for λ_B.

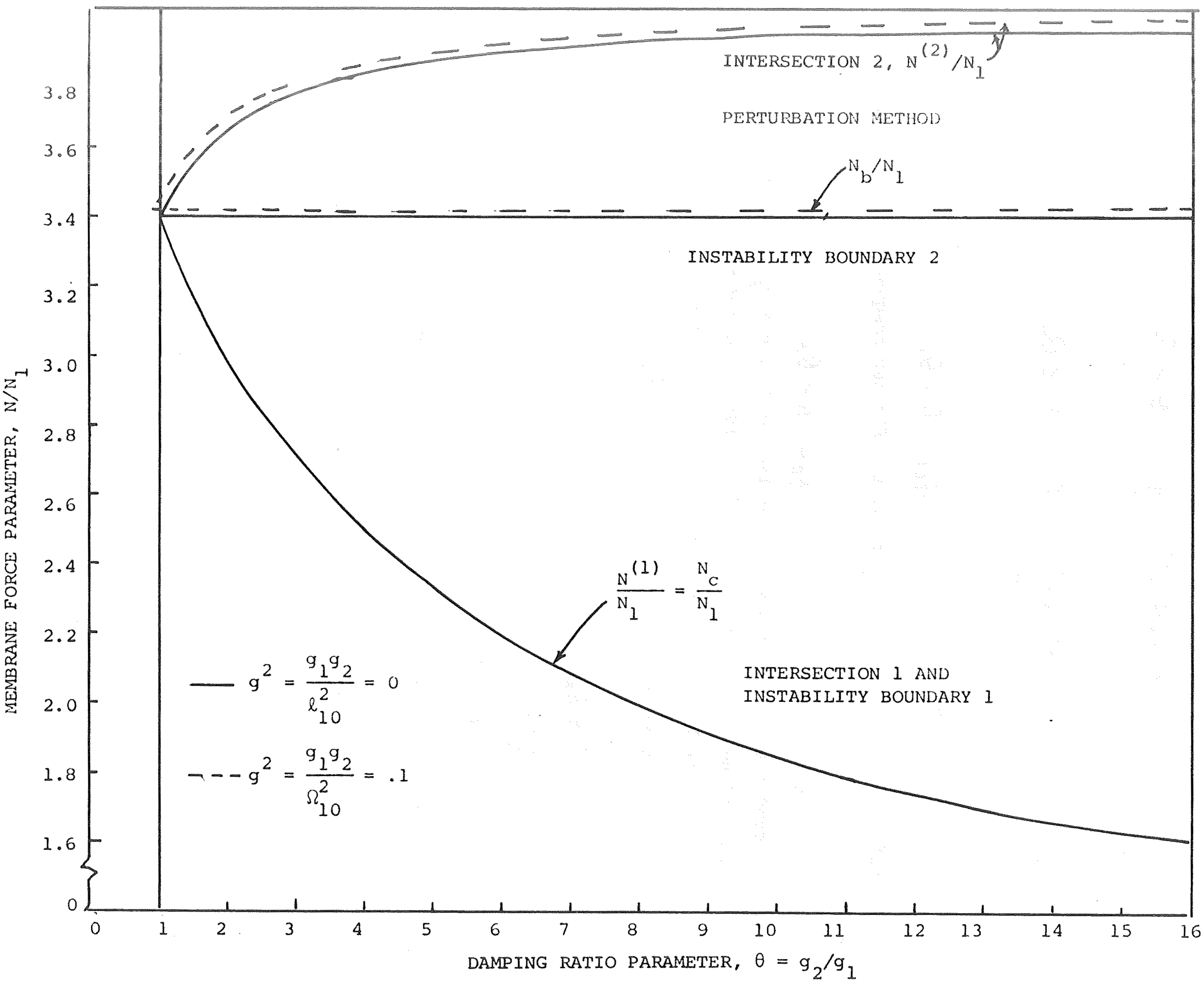


FIG. A.1 MEMBRANE FROCES N_b, N_c, $N^{(1)}$ AND $N^{(2)}$ AS FUNCTIONS OF THE DAMPING RATIO, $\theta = g_2/g_1$

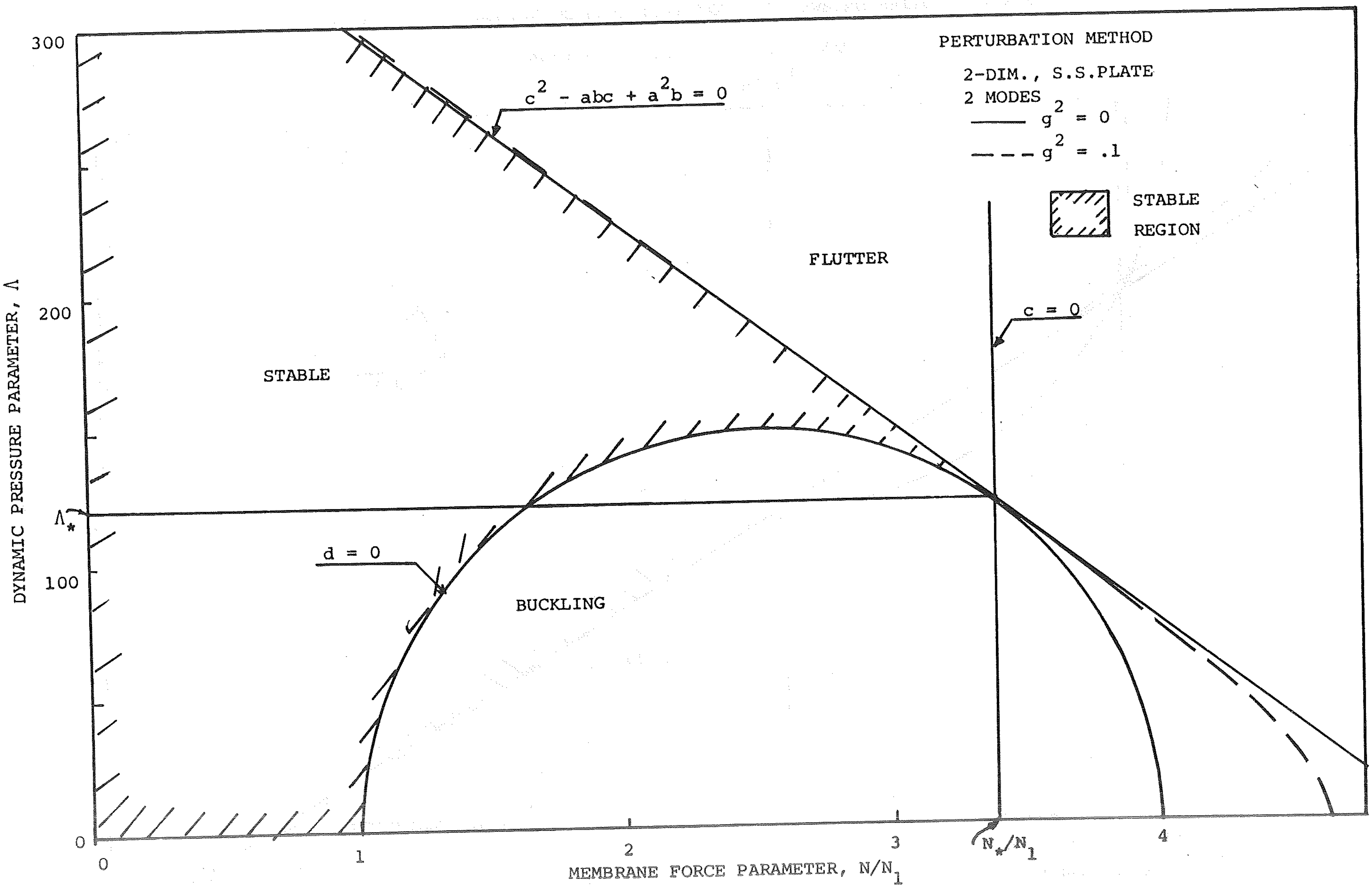


FIG. A.2 REGION OF STABILITY FOR THE DAMPING RATIO, $\theta = g_2/g_1 = 1$

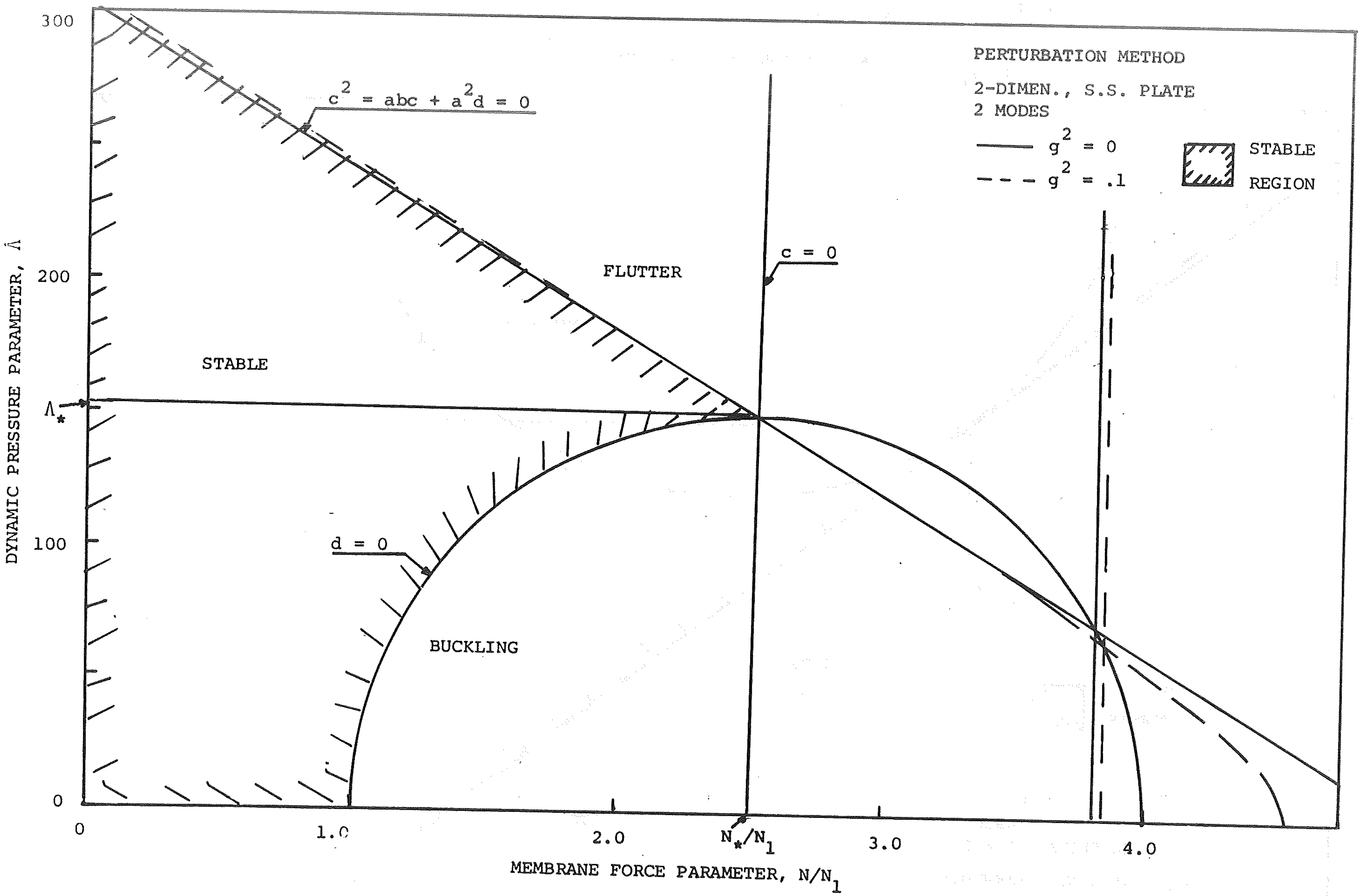


FIG. A.3 REGION OF STABILITY FOR THE DAMPING RATIO, $\theta = g_2/g_1 = 4$

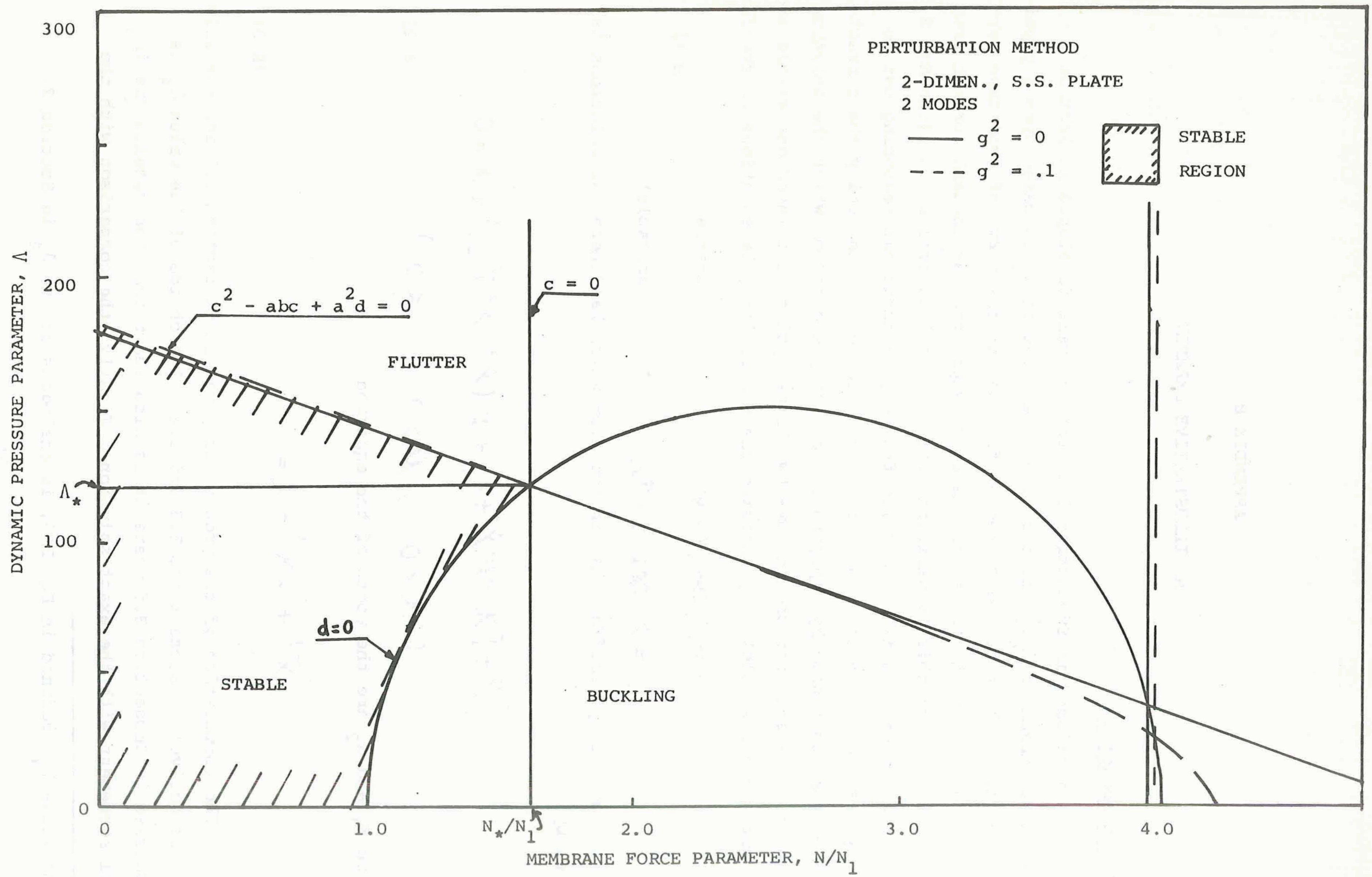


FIG. A.4 REGION OF STABILITY FOR THE DAMPING RATIO, $\theta = g_2/g_1 = 16$

APPENDIX B

AN ILLUSTRATIVE EXAMPLE

B.1 Introduction

As mentioned in Subsection 2.1, the analysis developed in Section 2 contains a parameter Λ_4. The choice of the value of Λ_4 is based upon a physical hypothesis which can be stated as follows: if the curve of amplitude versus Λ bends", then the "knee" of the curve separates the unstable branch from the stable branch; this "separation" point is used to evaluate Λ_4 (see Eq. 2.84). Since this hypothesis is not derived from any mathematical reasoning but is merely based upon physical intuition, it is convenient to verify the correctness of the assumption by applying it to a simple case for which the solution is known. The solution should have two limit cycles (the inner one stable and the outer one unstable). By choosing the two-limit-cycle solutions of the form

$$X = X_1 \cos(t+\phi_1) \quad \text{(stable)}$$
$$X = X_2 \cos(t+\phi_2) \quad \text{(unstable)} \tag{B.1}$$

it is immediately verified that an equation which has these two solutions is given by

$$\ddot{X} + [\lambda + \mu(\dot{X}^2+X^2) + \nu(\dot{X}^2+X^2)^2]\dot{X} + X = 0$$
$$(\lambda < 0 \quad , \mu > 0 \quad , \nu < 0) \tag{B.2}$$

where X_1 and X_2 are the roots of the equation

$$\nu\bar{X}^4 + \mu\bar{X}^2 + \lambda = 0 \tag{B.3}$$

The formulation of the problem is given in Subsection B.2 and the solution is derived in Subsections B.3 and B.4. The choice of the value Λ_4 is discussed in Subsection B.5 where it is also shown that the results are in full agreement with the exact solution.* Finally, the comparison with the

*The value λ_4, defined in Eq. B.4, is analogous to the Λ_4 in Section 2.

two-time-scaling technique (Refs. 8 and 9) is made in Subsection B.6 where it is shown that the two methods yield exactly the same results.

B.2 Formulation

Let

$$\lambda = -\varepsilon^2 + \lambda_4 \varepsilon^4 + O(\varepsilon^6) \tag{B.4}$$

and

$$X = \varepsilon X_1 + \varepsilon^3 X_3 + \varepsilon^5 X_5 + O(\varepsilon^7) \tag{B.5}$$

where the functions X_k depend upon the multiple time scales

$$t_0 = t, \quad t_2 = \varepsilon^2 t, \quad t_4 = \varepsilon^4 t, \quad \cdots \tag{B.6}$$

The scales t_k are treated as independent variables; hence

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^4 \frac{\partial}{\partial t_4} + O(\varepsilon^6) \tag{B.7}$$

Combining Eqs. B.2, B.4, B.5, and B.7, and separating terms of the same order, yields the following system

System Order ε:

$$\frac{\partial^2 X_1}{\partial t_0^2} + X_1 = 0 \tag{B.8}$$

System Order ε^3:

$$\frac{\partial^2 X_3}{\partial t_0^2} + X_3 = -2\frac{\partial^2 X_1}{\partial t_0 \partial t_2} + \frac{\partial X_1}{\partial t_0} - \mu \frac{\partial X_1}{\partial t_0}\left[\left(\frac{\partial X_1}{\partial t_0}\right)^2 + X_1^2\right] \tag{B.9}$$

System Order ε^5:

$$\frac{\partial^2 X_5}{\partial t_0^2} + X_5 = -\left(\frac{\partial^2 X_1}{\partial t_2^2} + 2\frac{\partial^2 X_1}{\partial t_0 \partial t_4} + 2\frac{\partial^2 X_3}{\partial t_0 \partial t_2} \right)$$

$$+ \left(\frac{\partial X_1}{\partial t_2} + \frac{\partial X_3}{\partial t_0} \right) - \lambda_4 \left(\frac{\partial X_1}{\partial t_0} \right)$$

$$- \mu \left\{ 2 \left[\frac{\partial X_1}{\partial t_0} \left(\frac{\partial X_1}{\partial t_2} + \frac{\partial X_3}{\partial t_0} \right) + X_1 X_3 \right] \frac{\partial X_1}{\partial t_0} \right.$$

$$\left. + \left[\left(\frac{\partial X_1}{\partial t_0} \right)^2 + X_1^2 \right] \left(\frac{\partial X_1}{\partial t_2} + \frac{\partial X_3}{\partial t_0} \right) \right\}$$

$$- \nu \left[\left(\frac{\partial X_1}{\partial t_0} \right)^2 + X_1^2 \right]^2 \frac{\partial X_1}{\partial t_0} \tag{B.10}$$

B.3 Solution

The solution of Eq. B.8 is given by

$$X_1 = A e^{it_0} + A^* e^{-it_0}$$

$$= 2\, Real\, (A e^{it_0}) \tag{B.11}$$

where A is the function of t_2, t_4, . . . and A* is the complex conjugate of A. Combining Eqs. B.9 and B.11 yields

$$\frac{\partial^2 X_3}{\partial t_0^2} + X_3 = -2\, Real \left[\left(2\frac{\partial A}{\partial t_2} - A + 4\mu A^2 A^* \right) i e^{it_0} \right] \tag{B.12}$$

In order to avoid secular terms, the condition

$$2\frac{\partial A}{\partial t_2} - A + 4\mu A^2 A^* = 0 \tag{B.13}$$

must be satisfied. This equation can be written in the form given by Eq. 2.26

with (note that here α, β, and γ are real numbers.)

$$\alpha = 2$$

$$\beta = -1/2$$

$$\gamma = 2\mu \tag{B.14}$$

The solution of Eq. B.13 is thus given by (see Eqs. 2.31 to 2.33)

$$A = |A| e^{i\phi} \tag{B.15}$$

with

$$|A| = (4\mu + K e^{-t_2})^{-1/2} \tag{B.16}$$

where ϕ and k are functions of t_4, t_6, . . . Since Eq. B.13 is now satisfied, the solution of Eq. B.12 is given by

$$X_3 = (B e^{it_0} + B^* e^{-it_0}) = 2\,\mathrm{Real}\,(B e^{it_0}) \tag{B.17}$$

Combining Eqs. B.4, B.11, and B.17 yields

$$X = 2\,\mathrm{Real}\left[(\varepsilon A + \varepsilon^3 B) e^{it_0}\right] + O(\varepsilon^5) \tag{B.18}$$

where A is given by Eq. B.15, and B is still undetermined. In order to determine the B as a function of t_2, the fifth-order equations, given by Eq. B.10, is considered in the next subsection.

B.4 The Functions $B(t_2)$ and $A(t_2, t_4)$

Combining Eqs. B.10, B.11, and B.16 yields

$$\frac{\partial^2 X_5}{\partial t_0^2} + X_5 = -2\,Real\,\Big\{ i\Big[2\frac{\partial B}{\partial t_2} - B + 4\mu\,(A^2B^* + 2AA^*B)$$

$$+ 2\frac{\partial A}{\partial t_4} + (\Lambda_4 + i\frac{1}{4})A - i\mu A^2A^*$$

$$+ 16\nu A^3A^{*2}\Big]e^{it_0} + [\mu A^3 - 4\mu^2A^4A^*]\,e^{i3t_0}\Big\} \tag{B.19}$$

In order to avoid secular terms, the condition

$$2\frac{\partial B}{\partial t_2} - B + 4\mu\,(A^2B^* + 2AA^*B) + 2\frac{\partial A}{\partial t_4}$$

$$+ (\lambda_4 + \frac{i}{4})A - i\mu A^2A^* + 16\nu A^3A^{*2} \doteq 0 \tag{B.20}$$

must be satisfied.* This equation can be written as

$$\frac{\partial B}{\partial t_2} + \beta B + \gamma\,(A^2B^* + 2AA^*B) + \frac{\delta}{\alpha} = 0 \tag{B.21}$$

with α, β, and γ given by Eq. B.14 and

$$\delta = \Big[\delta^{(0)} + \delta^{(1)}AA^* + \delta^{(2)}(AA^*)^2 + \frac{\partial}{\partial t_4}\ln A\Big]\alpha A \tag{B.22}$$

---

*Incidentally, if Eq. B.20 is satisfied, the solution of Eq. B.19 is given by

$$X_5 = 2\,Real\,\Big[Ce^{it_0} - \frac{\mu}{8}A^3(1 - 4\mu AA^*)e^{3it_0}\Big]$$

where C can be determined only by studying the seventh-order equation. Thus, the function X_5 is not of interest here; the aim of the analysis of Eq. B.20 is to obtain the functions $B(t_2)$ and $A(t_2, t_4)$.

with

$$\delta^{(0)} = \frac{1}{2}\lambda_4 + i\frac{1}{8}$$

$$\delta^{(1)} = -\frac{i\mu}{2}$$

$$\delta^{(2)} = 8\nu \tag{B.23}$$

Equations B.21 and B.22 are identical to Eqs. 2.55 and 2.56.

Following the same procedure outlined in Subsection 2.7 one obtains

$$B = (b_R + i\, b_I)A \tag{B.24}$$

with

$$b_R = \left[\left(\frac{1}{K}\frac{\partial K}{\partial t_4} + \frac{\mu}{\nu^2} - \lambda_4\right)t_2 + C_1\right]\left(\frac{1}{2} - 2\beta|A|^2\right)$$

$$-2\mu\left(\lambda_4 + \frac{\nu}{\mu^2}\right)|A|^2 - \frac{\nu}{\mu^2}K\, e^{-t_2}|A^2|\ln|A| \tag{B.25}$$

and

$$b_I = -\frac{\partial\phi}{\partial t_4}t_2 - \frac{1}{4}\ln|A| + C_2 \tag{B.26}$$

In order to avoid secular terms*, the conditions

$$\frac{1}{K}\frac{\partial K}{\partial t_4} + \frac{\mu}{\nu^2} - \lambda_4 = 0 \tag{B.27}$$

$$\frac{\partial\phi}{\partial t_4} = 0 \tag{B.28}$$

must be satisfied. Thus, $\phi = \phi_o$ is independent of t_4 (<u>no</u> change in frequency of order ε^4!), whereas

---

*As $t \to \infty$, for the stable limit cycles, or as $t \to -\infty$, for unstable limit cycles.

$$K = K_o e^{(\lambda_4 - \frac{\nu}{\mu^2})t_4} \tag{B.29}$$

Combining Eqs. B.16 and B.29 yields (see also Eq. B.6)

$$\begin{aligned} A &= \left\{ 4\mu + K_o e^{-t_2 + (\lambda_4 - \frac{\nu}{\mu^2})t_4} \right\}^{-1/2} e^{i\phi_o} \\ &= \left\{ 4\mu + K_o e^{[-\varepsilon^2 + \varepsilon^4(\lambda_4 - \frac{\nu}{\mu^2})]t} \right\}^{-1/2} e^{i\phi_o} \end{aligned} \tag{B.30}$$

where k_o and ϕ are functions of t_6, t_8 . . . The value of λ_4 is still undetermined. As was mentioned in Subsection B.1, the choice of λ_4 is based upon physical reasoning. This choice is discussed in the next subsection.

B.5 The Parameter λ_4 and the Final Expression for the Solution

As was mentioned in Subsection A.1, the main objective of this appendix is to verify the correctness of the assumption which determines the choice of the parameter λ_4 (which is analogous to Λ_4 in Section 2). A discussion of this assumption is given in Subsection 2.1, whereas the mathematical formulation is given in Subsection 2.5 (see Eqs. 2.79 and 2.81). The assumption can be summarized as follows: the "knee" of the curve $|A|$ versus λ separates the stable branch of the curve from the unstable one. Mathematically speaking, the value ε_{knee}, for which λ has its maximum value (see Eq. B.4) is

$$\varepsilon_{knee} = \sqrt{\frac{1}{2\lambda_4}} \tag{B.31}$$

and is equal to the value ε_{cr} for which the exponent of Eq. B.30 changes sign.

$$\varepsilon_{cr} = 1 / \sqrt{\lambda_4 - \nu/\mu^2} \tag{B.32}$$

The condition discussed above can be stated as

$$\varepsilon_{knee} = \varepsilon_{cr} \tag{B.33}$$

By combining Eqs. B.31, B.32, and B.33, one obtains

$$\lambda_4 = -\frac{\nu}{\mu^2} \tag{B.34}$$

and

$$\varepsilon_{knee} = \varepsilon_{cr.} = \sqrt{\frac{-\mu^2}{2\nu}} \tag{B.35}$$

Finally, the solution of the problem is given by (see Eq. B.18)

$$X = 2\,\mathrm{Real}\left[(\varepsilon A + \varepsilon^3 B)e^{it}\right] + O(\varepsilon^5) \tag{B.36}$$

where (see Eqs. B.15, B.30, and B.34) $A = |A|\ e^{i\phi}$ with

$$|A| = \left\{4\mu + K_0\, e^{-(1+\varepsilon^2 \frac{2\nu}{\mu^2})\varepsilon^2 t}\right\}^{-1/2} \tag{B.37}$$

and* (combining Eqs. B.24 to B.28, and Eq. B.34)

$$B = \left[-\frac{\nu}{\mu^2} K_1\, e^{-t_2} |A|^2 \ln|A| - i\frac{1}{4}\ \frac{\ln|A|}{(4\mu)^{1/2}}\right] A \tag{B.38}$$

The limit cycle solution, $X_{L.C.}$, is obtained by setting $t \to \infty$ for the stable limit cycle ($\varepsilon^2 < \mu^2/2\nu$) and $t \to -\infty$ for the unstable limit cycle ($\varepsilon^2 > \mu^2/2\nu$). In both cases one obtains

$$X_{L.C.} = 2\,\mathrm{Real}\left(\varepsilon \frac{1}{2\sqrt{\mu}}\, e^{i(t+\phi_0)}\right) + O(\varepsilon^5) \tag{B.39}$$

---

*Note that the arbitrary choice

$$c_1 = 0 \qquad\qquad c_2 = -\frac{1}{8}\ln 4\mu$$

has been made in Eqs. B.25 and B.26. This is made in order to have $B = 0$ in the limit cycle solution. Further exploration is needed in order to motivate this choice.

with (combining Eqs. B.4 and B.34)

$$\lambda = -\varepsilon^2 - \frac{\nu}{\mu^2}\varepsilon^4 \tag{B.40}$$

By setting $\varepsilon/\sqrt{\mu} = X$, Eqs. B.39 and B.40 reduce to

$$X_{L.C.} = X\cos(t+\phi_0) + O(\varepsilon^5) \tag{B.41}$$

and

$$\nu X^4 + \mu X^2 + \lambda = 0 \tag{B.42}$$

which is in full agreement with Eqs. B.2 and B.3.

B.6 Modified Two-Time-Scaling Technique

In order to compare the multiple-time-scaling technique with the two-time-scaling technique (Refs. 8 and 9), the analysis is repeated here by using the two-time-scaling technique. In order to maintain the same flexibility of the multiple-time-scaling, it is convenient to introduce stretching of both scales. This procedure (which generalizes the two-time-scaling technique) will be termed the "modified two-time-scaling technique". Thus, instead of the multiple scales (see Eq. B.6) consider the two "stretched" scales

$$\begin{aligned} t_0 &= (1+\varepsilon^4\sigma + \cdots)t \\ t_2 &= (\varepsilon^2+\varepsilon^4\tau+\cdots)t \end{aligned} \tag{B.43}$$

Then Eq. B.7 is replaced by

$$\begin{aligned} \frac{d}{dt} &= (1+\varepsilon^4\sigma+\cdots)\frac{\partial}{\partial t_0} + (\varepsilon^2+\varepsilon^4\tau+\cdots)\frac{\partial}{\partial t_2} \\ &= \frac{\partial}{\partial t_0} + \varepsilon^2\frac{\partial}{\partial t_2} + \varepsilon^4\left(\sigma\frac{\partial}{\partial t_0} + \tau\frac{\partial}{\partial t_2}\right) + O(\varepsilon^6) \end{aligned} \tag{B.44}$$

Thus, Eqs. B.8 to B.10 are still valid if $\partial/\partial t_4$ is replaced by $\sigma(\partial/\partial t_o) + \tau(\partial/\partial t_2)$. In other words, the only modification is that in Eq. B.10 the term $\partial^2 x_1/\partial t_4$ is modified as follows

$$\frac{\partial^2 X_1}{\partial t_0 \partial t_4} \longrightarrow \sigma \frac{\partial^2 X_1}{\partial t_0^2} + \tau \frac{\partial^2 X_1}{\partial t_0 \partial t_2} \tag{B.45}$$

Note that Eqs. B.8 and B.9 are equal in both methods. Thus, the results obtained in Subsection B.3 are valid for the modified-two-time-scaling method also. Hence by using Eqs. B.11, B.15, and B.16, one obtains

$$X_1 = 2\,Real\left[(4\mu + Ke^{-t_2})^{-1/2} e^{i(t_0+\phi)}\right] \tag{B.46}$$

where K and ϕ are functions of t_4 in the multiple-time-scaling technique, whereas they are constant in the modified-two-time-scaling technique. By using Eq. B.46, Eq. B.45 can be rewritten as

$$\left[-\frac{1}{2}(4\mu + Ke^{-t_2})^{-\frac{3}{2}} e^{-t_2}\frac{\partial K}{\partial t_4} + i(4\mu + Ke^{-t_2})^{-1/2}\frac{\partial \phi}{\partial t_4}\right]$$
$$\rightarrow \left[\frac{1}{2}\tau(4\mu + Ke^{-t_2})^{-3/2} Ke^{-t_2} + i\sigma(4\mu + Ke^{-t_2})^{-1/2}\right] \tag{B.47}$$

or separating real parts and imaginary parts

$$\frac{1}{K}\frac{\partial K}{\partial t_4} \longrightarrow -\tau$$
$$\frac{\partial \phi}{\partial t_4} \longrightarrow \sigma \tag{B.48}$$

Comparing Eqs. B.27, B.28, and B.48 yields

$$\tau = \lambda_4 - \frac{\mu}{\nu^2}$$
$$\sigma = 0 \tag{B.49}$$

Combining Eqs. B.15, B.16, B.43, and B.49 yields*

*In Eq. B.50, the symbols K_o and ϕ_o are used instead of K and ϕ as a reminder that K is considered a constant in the two-time-scaling technique.

$$A = \left\{ 4\mu + K_0 \, e^{[-\varepsilon^2 + \varepsilon^4 (\lambda_4 - \nu/\mu^2)]t} \right\}^{-1/2} e^{i\phi} \tag{B.50}$$

which is equal to Eq. B.30. The discussion of the value of λ_4 is the same in both cases. Thus, as mentioned in Subsection B.1, the multiple-time-scaling technique and the modified two-time-scaling technique yield exactly the same result.